Ph.D. Dissertation 4856

ON THE STRUCTURE AND CLASSIFICATION OF DIFFERENTIAL MANIFOLDS

by

DENNIS BARDEN

of

DOWNING COLLEGE

DISSERTATION SUBMITTED FOR THE DEGREE OF PH.D.

PREFACE

This thesis is original except in so far as is indicated either generally in the introduction to each chapter or specifically in the references throughout. In particular chapter two is independent of the work of B. Mazur on the same topic, though it is of course associated with it by the sharing of a common source of inspiration.

The main ideas on which the work is based stem from those of S. Smale. The principle, used in chapter I, of finding a cobordism with a simple handle structure is essentially that used by Smale in his paper 'On the structure of 5-manifolds', (reference [16] for chapter I) and was pointed out to me explicitly by C. T. C. Wall who also suggested certain techniques and drew my attention to related results([20] and [22]) of his own which I have used.

Chapter II is an extension of Smale's paper 'On the structure of manifolds' (reference [11] to chapter II) and I have used his approach, proving handle cancellation lemmas to obtain a theorem on minimal handle decompositions, from which several more specific results follow. The extension is achieved by proving stronger cancellation lemmas than did Smale, and the ideas for these follow naturally from the methods of Smale ([11]) and of H. Whitney ([15]) and from the descriptions of 'simple homotopy type torsion' in [2], [8] and [13].

Throughout I have availed myself of standard results of algebra and algebraic and differential topology, including in the last the question of rounding corners which I assume carried out wherever necessary. In particular in both chapters I have made important use of A. Haefliger's recent results, an extension of those of Whitney, on embeddings and isotopies of manifolds in manifolds.

Chapter I is to be published in the Annals of Mathematics. Chapter II has been submitted to the Proceedings of the London Mathematical Society.

INDEX

INTRODUCTION.

Considerable advances have been made recently in the study of differential manifolds. Notably the work of S. Smale on the structure of such manifolds has led to the proof of the Poincare conjecture and of the h-cobordism theorem for simply connected manifolds, both in dimensions at least five, and also to the classification, by Smale himself and by C. T. C. Wall, of certain higher dimensional manifolds.

Smale classified simply connected 5-manifolds whose second Stiefel-Whitney class vanishes, and in the first chapter of this thesis I extend this to all simply connected 5-manifolds, expressing each as the connected sum of indecomposable manifolds which are determined by its second homology group and an invariant derived from its second Stiefel-Whitney class. Using results of C. T. C. Wall the indecomposable manifolds will be so constructed that it is possible, for all 5-manifolds, to find precise embedding and immersion dimensions and to show the existence of minimal handle decompositions.

A theorem of A. Markov shows that this classification cannot be extended to all 5-manifolds but there seems no reason to doubt that it could be extended to each class of 5-manifolds with given fundamental group. Such a classification would require, among other things, an extension of Smale theory to manifolds which are not simply connected. This aspect of the problem is studied in chapter II.

Smale showed that manifolds could be regarded as formed from 'handles', one corresponding to each critical point of a non-degenerate function on the manifold, and deduced most of his results from a theorem, based on his 'handle-body theorem', minimising the number of handles used. I have followed the same plan but replaced the handle-body theorem by other handle cancellation lemmas, obtained by similar techniques to those of Smale, from which a minimal handle theorem may be deduced which, although not a complete generalisation of that of Smale, is sufficient to prove generalisations of his results.

The chapters are written independently with references given separately for each. In particular the classification in chapter I can be deduced from Smale's original theorems and does not use the more general theory given in the second chapter. However the notation is consistent throughout, coinciding largely with standard notation and, for handle theory, with that of Smale, except that '∂' is used in chapter I for boundaries and boundary homomorphisms whereas 'b' is used in chapter II. In chapter I 'b' is reserved for linking numbers.

I have restricted my attention to the category of C^{∞}-manifolds, though clearly the results extend to any other category in which the geometry is valid. For example since, by result of J. Cerf, any combinatorial 5-manifold has a differential structure and since the manifolds of chapter I are clearly indecomposable in any category the classification can be extended to combinatorial 5-manifolds.

CHAPTER I

SIMPLY CONNECTED FIVE-MANIFOLDS

S. Smale, using his theory of handlebodies, has classified under diffeomorphism closed, simply-connected, smooth 5-manifolds with vanishing second Stiefel-Whitney class. C.T.C. Wall has given in [23] a classification of (n-1)-connected (2n+1)-manifolds, which does not however cover the case n=2. The reason for incompleteness of the results in these lower dimensions is that the general theorems on embeddings, on the removal of intersections of submanifolds, and on the cancelling of handles are not proved in these cases and in many instances are known to be false. In this chapter special arguments, together with the general theorems so far as they are available, are used to complete the classification of closed simply-connected smooth 5-manifolds. A.A. Markov has proved that a general classification of 5-manifolds is impossible, but it seems reasonable that these methods, together with the generalisation of Smale theory to non simply-connected manifolds which is carried out in chapter two, should give some information in the case of 5-manifolds with a given non-trivial fundamental group.

Simply-connected 5-manifolds are put into a series of disjoint classes $\mathfrak{M}_j$, $j = -1, 0, 1, \ldots, \infty$, determined by their cobordism class and the precise nature of their second Stiefel-Whitney class w^2. In each class there is a canonical manifold X_j having the minimum homology consistent with its being in that class. The main theorem states that if M is any manifold in $\mathfrak{M}_j$, then $M = X_j \# M'$, where $\#$ denotes the connected sum and M' is in the class already classified by Smale, i.e. $w^2(M')=0$. The existence of X_j is guaranteed by the work of Wall on killing the middle homotopy groups of odd dimensional manifolds, however using the same author's work on the diffeomorphisms of 4-manifolds it is possible to give an explicit description. The uniqueness of X_j follows from the main theorem and the truth of the Poincare conjecture in dimension five.

The main step in the proof is to find a particular cobordism V_j, between X_j and M, which is simply connected, has the same second homology group as X_j and for which the inclusions of the boundary components X_j and M in V_j induce, respectively, an isomorphism and an epimorphism of second homology groups. It then follows from Smale's structure theory applied to the 6-dimensional V_j that it may be obtained from $X_j \times I$ by adding

3-handles with trivial attaching maps. Using the results of A. Haefliger on isotopies of submanifolds these attaching maps may be collected together in a 5-disc in $X_j \times (1)$, and the required decomposition is then clear.

To produce such a cobordism the methods of surgery of J. Milnor are employed. First π_1 and all except one non-zero element of H_2 are killed, and in the case of the non-trivial cobordism class this is already sufficient. However, in general $H_2(V_j)$ is now too small and it is necessary to recreate it. This is achieved by killing suitable elements in $H_3(V_j)$, and the rest of the proof is concerned with finding these elements and showing that, when they are removed by surgery, the elements produced in $H_2(V_j)$ are homologous to and have the same order as corresponding elements in $H_2(X_j)$ and in $H_2(M_j)$.

The explicit construction of the canonical manifolds is not needed for the proof of the decomposition theorem, but it produces minimal handle decompositions, and allows the calculation of exact embedding and immersion dimensions for all the manifolds. The nature of the invariants which determine the class $\mathfrak{M}_j$ of a manifold also allows some extension of the results.

In the preliminary paragraph some of the more important theorems which will be needed are recalled, some notation is introduced, and algebraic and number theoretic lemmas proved. In particular this paragraph contains the algebra necessary for the description of the invariants, and a discussion of the skew form b defined by linking numbers on the torsion subgroup of $H_2(M^5)$.

In the first paragraph the canonical manifolds X_j and the indecomposable manifolds M_k of Smale are constructed, and the classes of manifolds are defined. Some properties of the manifolds are deduced from their construction.

In paragraph 2 the main theorems are stated but not proved. A classification theorem is deduced and theorems on embeddings, immersions and decompositions of simply connected 5-manifolds are proved.

The proof of the main theorems, 2.2 and 2.3, occupies the remainder of the chapter. In the third paragraph the special cobordisms mentioned above are defined, the proof is outlined, and the first simplification of the cobordisms is performed. In the next the cobordisms are modified, rebuilding the second homology group, until they are suitable for the factoring lemma, 5.1, which is proved in the final paragraph. The proof is then completed.

I am grateful to C.T.C. Wall for drawing my attention to this problem, and for his helpful suggestions throughout. In particular I am indebted to him for giving me a draft of [20] before it was published, and for telling me the results of [22] before it was written.

0 PRELIMINARY RESULTS AND DEFINITIONS

We shall be concerned throughout with compact C^{∞}-manifolds M^n of dimension n, though non-compact manifolds may occur as submanifolds of compact ones. M, whose boundary ∂M need not be vacuous, will be assumed orientable with orientations, generators $[M, \partial M], [\partial M]$ respectively of $H_n(M, \partial M; Z)$ and $H_{n-1}(\partial M, Z)$, chosen such that $\partial [M, \partial M] = [\partial M]$, where ∂ is the homology boundary homomorphism. Most of the fundamental definitions and results required may be found in [21]. For the rounding of corners etc. see also [2].

0.1 There are several methods of combining two manifolds to form a third. $A \smile B$ is the disjoint union of the two manifolds A^n, B^n. If A and B have non-empty boundaries ∂A, ∂B, then $A+B$ is formed from $A \smile B$ by embedding $(n-1)$-discs in ∂A and ∂B, identifying them under an orientation reversing diffeomorphism (the orientations of the embedded discs being induced from those of ∂A and ∂B), and smoothing the corners. If ∂A and ∂B have diffeomorphic connected components A', B', then $A \smile_f B$ is obtained from $A \smile B$ by identifying these components under some specified diffeomorphism f, and rounding the corners. $A \# B$ (see [7],[9]) is formed from $A \cup B$ by embedding an n-disc in each, avoiding the boundaries, removing the interiors of these discs, identifying the bounding (n-1)-spheres of the resulting holes under an orientation reversing diffeomorphism and rounding off the corners. Note that $\partial(A+B) = \partial A \# \partial B$, the $\#$ taking place between the components of ∂A and ∂B on which the $+$ was effected.

0.2 The manifold $A^n + h^r$, described as 'A with an r-handle attached' is formed from $A \cup D^r \times D^{n-r}$ by identifying $S^{r-1} \times S^{n-r}$ with its image under some embedding in ∂A, and rounding any corners. A decomposition of A on M is the presentation of A^n as $M^n + h_1^o + \ldots + h_{i_o}^o + \ldots + h_{i_n}^n$. Usually M is D^n or $Q^{n-1} \times I$ for some union Q of connected components of ∂A. Smale calls manifolds $D^n + h_1^r + \ldots + h_k^r$ handlebodies and denotes the set of all such for fixed n, r, and k by $\mathcal{H}(n; r, k)$. The following theorem is due to him ([15]).

Theorem A If $n > 5$ and M^n is compact and simply-connected, all its connected boundary components are simply-connected, $\partial M = Q_1 \cup Q_2$ where each Q_i is the union of connected components, and $H_k(M, Q_1; Z)$ has a direct sum decomposition with β_k infinite cyclic summands and σ_k finite cyclic summands, then M has a decomposition on $Q_1 \times I$ with $\beta_k + \sigma_k + \sigma_{k-1}$ k-handles for each k. (If n=5 all that can be stated under similar hypotheses is that a decomposition may be found without 1- or 4-handles and no more 0- or 5-handles than are required to satisfy the relation between Q_1, Q_2, and the connected components of M. This follows from the methods used by Smale, and is also proved explicitly by A.H. Wallace in [25].)

There is a close connection between decompositions of a manifold M and 'nice functions' on it, that is non-degenerate differential functions $f: M \longrightarrow R^1$, where R^1 denotes the real line, which are transverse to ∂M in a neighbourhood of it and satisfy $f(Q_1) = -{}^1/2$ $f(Q_2) = n + {}^1/2$. To each such function correspond decompositions of M on $Q_1 \times I$ with exactly one k-handle for each critical point of f of index k. Conversely for each decomposition there are nice functions with the corresponding number of critical points.

If A has a decomposition on $\partial A \times I$ with α_i i-handles $i=0,1 \ldots, n$ and B has one on $\partial B \times I$ with β_i i-handles where $\partial A \cong \partial B$ (diffeomorphic), then $M = A \cup_f B$ has a decomposition (on D^n) with $\alpha_i + \beta_{n-i}$ i-handles. For if $f: A \longrightarrow R$ and $g: B \longrightarrow R$ correspond to the given decompositions with constants adjusted so that $f(\partial A) = -g(\partial B)$ then $h: M \longrightarrow R$ defined by $h(A) = f(A)$, $h(B) = -g(B)$ has precisely $\alpha_i + \beta_{n-i}$ critical points since none are introduced along $\partial A = \partial B$.

0.3 Theorem B (Haefliger [4])

If V^v is a closed v-dimensional C^∞-manifold and M^m an m-dimensional C^∞-manifold,

(a) if $f: V \longrightarrow M$ is a continuous map with $\pi_i(f) = 0$ for $i < 2v-m+2$ then f is homotopic to an embedding provided $2m > 3v+2$,

(b) if $f, g; V \longrightarrow M$ are homotopic embeddings with $\pi_i(f) = \pi_i(g) = 0$ for $i < 2v-m+3$ then g is isotopic to f provided $2m > 3v+3$.

Here $\pi_i(f)$ denotes $\pi_i(C_f, V)$ where C_f, the mapping cylinder of f, is formed from $V \times [0,1] \cup M$ by identifying, for each v in V, $v \times [1]$ with $f(v)$. $V \times [0]$ in C_f is referred to as V.

From this theorem it follows that two 2-spheres embedded in a simply connected 5-manifold are isotopic if and only if they are homotopic (if and only if they are homologous), and that any continuous map of a 3-sphere into a simply connected 6-manifold may be approximated by an embedding.

0.4 In an orientable n-manifold w^2 is the obstruction to parallelisability over the two-skeleton, since it is the obstruction to the existence of an (n-1)-field over the two-skeleton ([18]) and the complementary 1-frame must be continuous.

In a simply connected manifold M, $w^2(M) \in H^2(M ; Z_2) = H_2(M) \pitchfork Z_2$ † may be regarded as a homomorphism $w^2 ; H_2(M) \longrightarrow Z_2$.

In a 6-manifold the obstruction to s-parallelisability over an embedded 2-sphere is the obstruction to triviality of its normal bundle. For $\tau(M) \mid S^2 = \tau(S^2) + \nu(S^2 \subset M)$ and so $\tau(M) \mid S^2 + \varepsilon^1 = \varepsilon^7 \Longleftrightarrow \nu(S^2 \subset M) + \varepsilon^3 = \varepsilon^7$, i.e. $\Longleftrightarrow \nu(S^2 \subset M)$ is stably trivial; but it is already stable. Thus in a simply connected (orientable) 6-manifold the value of w^2 on the homology class carried by an embedded 2-sphere is the obstruction to the triviality of the normal bundle of this sphere.

For orientable 5-manifolds $w^2.w^3[M]$ is the only possibly non-zero Stiefel Whitney number, and thus the oriented cobordism group, $\Omega_5 = Z_2$ (see [19]). w^3 is the mod 2 reduction of the integer class $W^3 = \delta^* w^2$ where δ^* is the Bockstein associated with the coefficient sequence

$$0 \longrightarrow Z \xrightarrow{2} Z \longrightarrow Z_2 \longrightarrow 0$$

---

† Throughout this chapter, if A and B are abelian groups we shall denote by $A \pitchfork B$ the group of homomorphisms of A into B.

0.5 In an abelian group G a set of r non-zero elements will be termed independent if the subgroup which they generate together is the direct sum of the r cyclic subgroups which they generate separately. By a basis of a finitely generated abelian group G shall be meant an independent set which generates G. The set of non-zero elements $x_1, \ldots x_r$ is independent if and only if $n_1x_1 + \ldots + n_rx_r = 0$ implies $n_ix_i = 0$ for each i. Any maximal pure independent set forms a basis (see [6]).

Clearly the number of elements in a basis is not in general an invariant of G. However the number of elements of infinite order is invariant, and for the torsion subgroup the orders and the number of elements in a basis with the most or the fewest possible elements are invariants. In the first case the number of elements of order p^i is the ith Ulm invariant of the p-primary component of G, and in the second case we have the classical decomposition of a finite abelian group as a direct sum $Z_{k_1} + Z_{k_2} + \ldots + Z_{k_r}$ where k_i divides k_{i+1}. A basis with the most possible elements will be called a U-basis.

Lemma C If A is a finitely generated abelian group and w, $A \longrightarrow Z_p$ a homomorphism into the cyclic group of order p, then there is a U-basis of A such that w is zero on all elements of this basis except possibly one. If this element has order p^i, $0 \leq i \leq \infty$, then $i = i(w) = i(w \circ \alpha)$ for any α in Aut(A).

<u>Proof</u> Let e_1, e_2 be elements of a U-basis $(e_1, \ldots e_r)$, of orders p^{k_i} with $k_1 \leq k_2$ such that $w(e_1) = u$, $w(e_2) = ku$ for some u in Z_p. Then the set $(e_2-ke_1, e_1, e_3, \ldots e_r)$ is clearly a U-basis and $w(e_2-ke_1)=0$. (The set clearly spans A and $gp(e_1,e_2) = gp(e_1,e_2-ke_1) = gp(e_1) + gp(e_2-ke_1)$, since if $n_1e_1 + n_2(e_2-ke_1) = 0$ then p^{k_2} divides n_2 and so also p^{k_1} divides n_2 and thence n_1.)

Since the basis is finite, such changes will eventually produce a U-basis having the required properties. The possibility $k_i = \infty$ is not excluded from the above.

If w is not the zero homomorphism then in every basis there is at least one element on which w is non-zero. If $(e, e_1, \ldots e_r)$ is a basis such that $w(e) \neq 0$, $w(e_i) = 0$, $i=1, \ldots r$, and e is of infinite order, then w is zero on the torsion subgroup of A. For it is non-zero only on elements $f = ne+n_1e_1 + \ldots + n_re_r$, where $n \neq 0$, and order f cannot be finite since e and the e_i are independent.

Since w is necessarily zero on q-primary components for $q \neq p$, to show that $i(w)$ is well defined it is thus sufficient to consider A_p, the p-primary component of A. Let $f_1, f_2, \ldots f_r$ be a basis (necessarily a U-basis) of A_p such that $w(f_1) \neq 0$, $w(f_i) = 0$ for $i > 1$. Let e be any element of A_p with order less than that of f_1 then $e = n_1 f_1 + n_2 f_2 + \ldots + n_r f_r$ has order less than that of f_1 which, as the f_i are independent means that p divides n_1 and so $w(e_1) = 0$. Thus if $w(e_1) \neq 0$ the order of e is at least as great as that of f_1, and if e is the only element of some basis which is non-zero under w its order must be precisely that of f_1.

The image of a U-basis under an automorphism α of A is another U-basis. For independent elements map to independent elements since α is monomorphic and the images of the basis elements generate A since α is epimorphic, clearly the new basis must also be a U-basis. Thus the above argument also shows that $i(w) = i(w \circ \alpha)$.

Corollary If M is a 1-connected 5-manifold, $i(w^2(M))$ is a diffeomorphism invariant of M.

Proof As remarked in 0.4 above, w^2 may be regarded as a homomorphism $w: H_2(M) \longrightarrow Z_2$. $H_2(M)$ is a finitely generated abelian group so by the lemma $i(M) = i(w^2(M))$ is defined. A diffeomorphism of M induces an automorphism of $H_2(M)$ so that $i(M)$ is indeed a diffeomorphism invariant.

0.6 There are two 3-disc bundles over the 2-sphere since $\pi_1(SO_3) = Z_2$; denote these A,B where A is the trivial bundle $D^3 \times S^2$. Thus $\partial A = S^2 \times S^2$, $\partial B = P \# Q$ where P denotes the complex projective plane and Q is the same space with the opposite orientation. $H_2(A) = H_2(B) = Z$, $w^2(A) = 0$, $w^2(B) \neq 0$. $H_2(\partial A) = Z+Z = H_2(\partial B)$ and we assume generators chosen once for all as follows; choose generators u of $H_2(A)$, v of $H_2(B)$ then choose generators a,b, of $H_2(\partial A)$ corresponding to the factor S^2's, and generators p,q of $H_2(\partial B)$ corresponding to the summands P,Q, such that $i_* a=u$, $i_* b=0$, $i_* p=v=i_* q$, where i denotes the relevant inclusion map.† (For details see [18])

If '.' denote the intersection number of homology classes then a.b=1, p.p=1, q.q= -1, and a.a=b.b=p.q = 0. Since $S^2 = D^2_N \cup_{id} D^2_S$, the union of its northern and southern hemispheres, and since any bundle over a disc is trivial both A and B are of the form $D^5 \cup D^5$ with identification along certain $D^3 \times S^1$'s in the bounding S^4 of each disc, i.e. they are of the form $D^5 + h^2$. Conversely any manifold of $\mathcal{H}(5; 2,1)$ is one or other of these disc bundles. (C.f. [14], [16])

† 'i' will be reserved for inclusion maps throughout this chapter, and it will not be stated which it represents if this is clear from the context.

If M is a simply connected 5-manifold with $A \subset M$ or $B \subset M$ then $w^2(i_*u)=0$ $w^2(i_*v)\neq 0$ on account of the corresponding properties of u,v and the interpretation of w^2 as the obstruction to parallelisability over the 2-skeleton.

0.7 Linking numbers in an n-manifold M^n are defined between the torsion subgroups of $H_p(M)$ and $H_q(M)$ whenever $p+q=n-1$ ([13], [7] and [20]). Consider the commutative diagram (1), where the rows are part of the exact sequences induced by the coefficient sequence

$$0 \longrightarrow Z \longrightarrow Q \longrightarrow Q/Z \longrightarrow 0,$$

D are duality isomorphisms, and U are isomorphisms following from the universal coefficient theorem since Q and Q/Z are divisible.

(1)
$$\begin{tikzcd}
H_q(M) \otimes Q \arrow[r, "\gamma"] \arrow[d, dashed, "\cong"', "U"] & H_q(M) \otimes Q/Z \arrow[d, dashed, "\cong"', "U"] & & \\
H^q(M,Q) \arrow[r] \arrow[d, dashed, "\cong"', "D"] & H^q(M,Q/Z) \arrow[d, dashed, "\cong"', "D"] & & \\
H_{p+1}(M,Q) \arrow[r, "\beta"] & H_{p+1}(M,Q/Z) \arrow[r] & H_p(M,Z) \arrow[r, "\alpha"] & H_p(M,Q)
\end{tikzcd}$$

Now torsion $(H_p(M,Z)) \cong \ker\alpha, \cong \operatorname{coker}\beta$ by exactness, and by $U^{-1}\circ D^{-1}$ $\cong \operatorname{coker}\gamma$. So that any $x \in \text{tors}\,(H_p(M,Z))$ determines a coset in $H_q(M,Z) \otimes Q/Z$ all elements of which have the same value on any $y \in \text{tors}\,(H_q(M,Z))$. We denote this value

$b(x,y)$, the linking number of x with y. It is a rational number modulo 1 which satisfies $b(x,y) + (-1)^{pq}b(y,x) = 0$. In particular if M is a $(4k+1)$-dimensional manifold b gives a skew symmetric non-singular bilinear form on tors $(H_{2k}(M))$. In [20] Wall shows that this form determines completely the possibility of killing the middle homotopy groups of odd dimensional manifolds. (On $(4k+3)$-dimensional manifolds the form is symmetric.) It is also relevant to the methods and results of the present paper, and in particular we require the following number theoretic lemma which occurs (lemma 4 (ii)) in [20].

Lemma D Let $b: G \otimes G \longrightarrow Q/Z$ be a non-singular bilinear skew form on the finite abelian group G, then there exist elements x_i, y_i of order θ_i in G such that $b(x_i,x_j) = b(x_i,y_j) = b(y_i,y_j) = 0$ for $i \neq j$, $b(x_i,x_i) = 0$ and $b(x_i,y_i) = 1/\theta_i$. The direct sum of the cyclic subgroups generated by the x_i and the y_i is a direct summand of G of index at most 2. We shall call a generator of the other direct summand z, understanding this to be zero when no such summand occurs. $b(x_i,z) = 0 = b(y_i,z)$ for all i.

Corollary Tors $(H_{2k}(M^{4k+1})) = B+B$ or $B+B+Z_2$.

Remark In [20] the decomposition is carried out separately on each p-primary component so that the basis (x_i, y_i, z) is a U-basis. However this is not necessary for let x_1, y_1 have order m, x_2, y_2 have order n, where $(m,n)=1$ and $b(x_i,x_j)=0=b(x_i,y_j)$ for $i \neq j$ $b(y_1,y_2)=0$, $b(x_1,y_1)=1/m$ and $b(x_2,y_2)=1/n$. Then $x_1 + x_2$, $y_1 + y_2$ have order mn and form a basis of $gp(x_1,x_2,y_1,y_2)$. Moreover $b(x_1 + x_2, x_1 + x_2) = 0$ and $b(x_1 + x_2, y_1 + y_2) = (m+n)/mn$. But $(m+n,mn)=1$ so there exist k,q such that $k(m+n) + q(mn) = 1$, and then $k(x_1 + x_2)$, $y_1 + y_2$ are generators with linking number $k(m+n)/mn = 1/mn$ modulo 1. Similarly if θ is odd, z,x,y may be replaced by $z_1, z_2 = z+x, y$.

Definition A basis $(x_i, y_i, z_1, z_2, e_i)$ of G, where the e_i are of infinite order and the remaining elements as in the lemma or the remark, will be called a b-basis of G.

From his main theorem together with knowledge of the Wu manifold (X_{-1} of §1 of the present chapter) Wall deduces (Propositions 1 and 2 of [20]).

Lemma E For 1-connected 5-manifolds M^5, $b(x,x)\neq 0 \iff w^2(x)\neq 0$; and $z\neq 0$, i.e. the extra Z_2 appears in the corollary to lemma D, if and only if M is in the non-trivial cobordism class.

That $M \not\sim 0$ implies $H_2(M)$ has a summand Z_2 also follows from corollary 3.3 below, this summand corresponding to the occurrence of the Wu manifold in the canonical decomposition of M.

The reduction of $H_2(M)$ under b given by Lemma D may be extended to include that of $b(x,x)$ corresponding to the reduction of $w^2(x)$ in Lemma C:

<u>Complement to Lemma D</u> Let M be a simply connected 5-manifold and b the skew form on $H_2(M)$ defined by linking numbers, then there is a b-basis of $H_2(M)$ which is also a U-basis such that $w^2(M)$ is non-zero on at most one of its elements.

<u>Proof</u> The basis elements of infinite order may be changed as in the proof of lemma C, since their belonging to a b-basis imposes no restriction on them. For the elements of finite order we have by lemma E $b(x,x)\neq 0 \Longleftrightarrow w^2(x)\neq 0$. Let (x_i,y_i,z,e_j) be a b-basis of $H_2(M)$ which is also a U-basis, and let $b(y_1,y_1)=1/2=b(y_2,y_2)$. Then 2 divides θ_i, the order of y_i, and, as these are elements of a U-basis, θ_i must be a power of 2. Thus if $\theta_1 \leq \theta_2$, θ_1 divides θ_2 and we may define an automorphism of $H_2(M)$ by $x_1,x_2 \longrightarrow x_1-(\theta_2/\theta_1)x_2$, x_2 and $y_1,y_2 \longrightarrow y_1$, $y_1+y_2+(\theta_1/2)x_1$, the other basis elements being left fixed. This gives another U-basis which is still a b-basis but with one fewer element such that $b(x,x)\neq 0$, i.e. such that $w^2(x)\neq 0$. Repeating this as

often as possible, together with a final step replacing y_1 by y_1+z (if $z\neq 0$ then since b is non-singular $b(z,z)\neq 0$) we obtain the required basis.

As in the remark following lemma D the basis elements may now be recombined, in particular to give a b-basis with the minimum number of elements such that the one element, on which w^2 is non-zero has order a power of 2 (perhaps 1 or ∞). Such a basis corresponds to the canonical decomposition of M which will be given in 4, there being one factor, the Wu manifold, if $z\neq 0$, one for each pair x_i, y_i and one for each e_i. These remaining factors are all manifolds used by Smale in [16] except that if $w^2(e_1)\neq 0$ the corresponding factor is the non-trivial 3-sphere bundle over the 2-sphere, or if $w^2(y_1)\neq 0$ it is a specially constructed manifold depending on the order of y_1. More generally it will be shown that to any b-basis of $H_2(M)$ there corresponds, in a similar manner, a decomposition of M itself.

<u>0.8</u> The following lemma is due to Smale (lemma 1.3 of [16]).

<u>Lemma F</u> The set of simply connected 5-manifolds with zero w^2 coincides with the set $\partial\mathcal{H}(6;3,k)$.

<u>Proof</u> $\bar{w}^2 = w^2 = 0$ so the normal bundle of $M^5 \subset R^{12}$ has a 6-frame field over the 2-skeleton. The

obstructions to extending this over higher skeletons are in $H^r(M,\pi_{r-1}(V_{7,6}))$ which are all zero since $\pi_2(V_{7,6}) = \pi_4(V_{7,6}) = 0$ (see [5]). Thus the normal 7-frame bundle has a cross-section and the Thom class $\in \pi_{12}(S^7) = 0$ may be defined. Since it is zero $M = \partial W$ with W s-parallelisable and, after surgery, 2-connected (c.f. Milnor [9]). Then by the homology sequence of (W,M) and duality it follows that $H_r(W)=0$ except for $r=0,3$. So $W \in \mathcal{H}(6;3,k)$ by theorem A. Conversely every $W \in \mathcal{H}(6;3,k)$ is parallelisable since obstructions have coefficients in $\pi_2(SO_4)=0$. Hence ∂W is s-parallelisable and so $w^2(\partial W)=0$. ∂W is obtained from S^5 removing 2-spheres by surgery; this does not affect π_1 and so $\pi_1(\partial W)=1$.

1. THE MANIFOLDS

For the notation used below see 0.1, 0.6. Generators of the second homology groups of various copies of the disc bundles A,B will carry the same suffixes as the bundles.

Certain manifolds will be constructed in the form $M^5 \cup {}_f M^5$, for which it is necessary to obtain suitable diffeomorphisms f realising given automorphisms $f_*: H_2(\partial M) \longrightarrow H_2(\partial M)$. This problem has been studied by C.T.C. Wall in [22] where he shows that a large class of automorphisms of $H_2(M^4)$ can be realised by diffeomorphisms. This class certainly contains the automorphisms which we want to realise in the particular 4-manifolds considered below.

Construction

$M_1 = S^5$, $M_\infty = S^2 \times S^3$

For $1 < k < \infty$ $M_k = (A_1^* + A_2^*) \cup f_k \ (A_1 + A_2)$, where f_k realises the isomorphism given by $(a_1^*, b_1^*, a_2^*, b_2^*) = (a_1, b_1, a_2, b_2)\underline{A}(k)$ the matrix $\underline{A}(k)$ being $\begin{pmatrix} 1 & 0 & 0 & -k \\ 0 & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Note that

$\underline{A}(k)\underline{A}(1) = \underline{A}(k+1)$.

$X_{-1} = B^* \cup g_{-1}B$, $X_\infty = B^* \cup g_\infty B$, where g_{-1} induces $(p^*, q^*) = (p, -q)$ and g_∞ induces $(p^*, q^*) = (p, q)$.

i.e. g_∞ could be the 'identity' diffeomorphism between the two copies of $P \neq Q$.

For $0 < j < \infty, X_j = (B_1^* + B_2^*) \cup g_j \ (B_1 + B_2)$.

g_1 induces the isomorphism $(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, -q_1, p_2, -q_2)$.

For $j > 1$ g_j induces $(p_1^*, q_1^*, p_2^*, q_2^*) = (p_1, q_1, p_2, q_2)\ \underline{B}(2^{j-2})$,

where $\underline{B}(n)$ is the matrix $\begin{pmatrix} 1 & 0 & n & -n \\ 0 & 1 & n & -n \\ -n & n & 1 & 0 \\ -n & n & 0 & 1 \end{pmatrix}$. Here again we have

the inductive construction $\underline{B}(n)\ \ \underline{B}(1) = \underline{B}(n+1)$.

<u>Lemma 1.1</u> All the manifolds are simply-connected and

(i) $H_2(M_k) = Z_k + Z_k$ for $k \neq 1, \infty$

(ii) $H_2(X_{-1}) = Z_2$, $H_2(X_\infty) = Z$, $H_2(M_\infty) = Z$

(iii) $H_2(X_j) = Z_2 j + Z_2 j$ for $0 < j < \infty$

(iv) $w^2(M_k) = 0$ for all k.

(v) $w^2(X_j) \neq 0$ for any j.

<u>Proof</u> (i) Generators for $H_2(M_k)$ are carried by the inclusions of u_1 and u_2. This can be seen for example from the decomposition of M obtained from those of $A_1 + A_2$ and $A_1^* + A_2^*$ as in 0.2. In this the only 2-handles are those corresponding to A_1, A_2. There are relations $b_1^* = ka_2 + b_1$, $b_2^* = -ka_1 + b_2$ which, after inclusion in M yield $0 = i(b_1^*) = k.i(u_2) + i(b_1) = k.i(u_2)$, since $i(b_1)$ is already zero in $A_1 + A_2$ and so a fortiori in M.

Similarly $k.i(u_1) = 0$. The relations $a_1^* = a_1$, $a_2^* = a_2$ are redundent and, there are clearly no others (i) is proved.

(ii) and (iii) follow by a similar argument. e.g. for $1 < j < \infty$ the relations $p_1^* = p_1 - 2^{j-2}p_2 - 2^{j-2}q_2$ and $q_1^* = q_1 + 2^{j-2}p_2 + 2^{j-2}q_2$ lead to $i(v_1) - i(2^{j-1}v_2) =$ $i(p_1) - 2^{j-2}i(p_2) - 2^{j-2}i(q_2) = i(p_1^*) = i(q_1^*) = i(v_1) + i(2^{j-1}v_2)$, whence $2^j.i(v_2) = 0$. The other two relations lead to $2^j.i(v_1) = 0$ giving the required second homology group.

(iv) follows from 0.4 since, by the handle decompositions of M_k mentioned above, they are 1-connected and generators of their second homology groups are carried by copies of the trivial 3-disc bundle A.

(v) is similar. In particular $w^2(i(v_1)) \neq 0$.

<u>Remark</u> (1) The manifolds M_k are those also called M_k by Smale in [16]. This follows from lemma 1.1 and Smale's classification theorem.

(2) X_∞ is the nontrivial 3-sphere bundle over the 2-sphere. This can be seen by pulling the latter apart as in Steenrod [18]. (Since $\pi_1(SO_3) \longrightarrow \pi_1(SO_4)$ is epimorphic and SO_3 maps each D_i into itself where $S^3 = D_1 \cup_{id} D_2$, the sphere bundle is the union of the subbundles obtained by restricting the fibre to these

discs.) X_{-1} is the Wu manifold (c.f.[3]).

(3) $X_1 = X_{-1} \# X_{-1}$, as is clear from their definitions, but otherwise X_j, and M_k if $k=1,\infty$ or a prime power, are not decomposable. Except for M_2 this follows from lemma D. However if M_2 were decomposable one factor M' would have to have $H_2(M')=Z_2, w^2(M')=0$. By the Wall cobordism criterion, lemma E, this is impossible. (This could also be seen from lemma F, for if $M'=\partial H$, $H \in \mathcal{H}(6;3,k)$, then $i: H_3(H) \longrightarrow H_3(H,M')$ is a monomorphism of free groups, determined by the skew intersection matrix, so its cokernel is not Z_2 (c.f.[9],[16]).)

(4) The bases of $H_2(M_k)$, $H_2(X_j)$ for $k \neq 1,\infty$, $j \neq -1,\infty$ used above are not those used in §4 in the proofs of the theorems. There we replace u_1,u_2 by u_1,u_1+u_2, and similarly v_1,v_2. This gives the basis of lemma D.

<u>Lemma 1.2</u> If $W^3(M) \in H^3(M,Z)$ denote the integral third Stiefel-Whitney class of M and $w^3(M) \in H^3(M,Z_2)$ its mod two reduction then

(i) $W^3(X_j)$ is nonzero except when $j = \infty$

(ii) $w^3(X_j)$ is nonzero only when $j = -1,1$.

<u>Proof</u> $W^3 = \delta w^2$ and as w^2 is known it is sufficient to calculate δ. Corresponding to the decompositions of X_j derived from their construction, the X_j, for $j \neq -1,\infty$, have cell decompositions with cells $e^0, e_1^2, e_2^2, e_1^3, e_2^3, e^5$.

Denote (e_i) the integer chain carried by e_i, $[e_i]$ the dual cochain and $[e_i]_2$ its mod two reduction. Thus (e_i^2) represent the generators of $H_2(X_j,Z)$. Using the methods of Smale for adding handles, i.e. replacing 3-handles whose attaching maps carry homology elements x,y by handles attached by maps carrying x,x+y, it may be assumed that the 3-handles, and so also the 3-cells, directly induce the relations in $H_2(X_j)$, i.e. that $\partial(e_i^3)\sim 2^j(e_i^2)$ for each i. There are thus no boundaries in $C_2(X,\mathbf{Z})$ and so $\partial(e_i^3) = 2^j(e_i^2)$. Now (e_i^3) is a cycle mod 2 carrying a generator of $H_3(X_j,Z_2)$. The Bockstein is calculated from

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^2(X,Z) & \xrightarrow{\times 2} & C^2(X,Z) & \longrightarrow & C^2(X,Z_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & C^3(X,Z) & \xrightarrow{\times 2} & C^3(X,Z) & \longrightarrow & C^3(X,Z_2) & \longrightarrow & 0
\end{array}$$

$[e_i^2]_2$ lifts to $[e_i^2]$ in $C^2(X,Z)$ and, as $\partial(e_i^3) = 2^j(e_i^2)$, $\delta[e_i^2] = 2^j[e_i^e] = (2^{j-1}[e_i^3])\times 2$. So if $\{\ \}$ denotes cohomology class $\delta^*\{[e_i^2]_2\} = 2^{j-1}\{[e_i^3]\}$. However the 2-cells e_i^2 correspond to the non-trivial bundles B_i, so $w^2(X) = \{[e_1^2]_2\} + \{[e_2^2]_2\}$ and $W^3(X_j) = \delta^* w^2(X_j) = 2^{j-1}\{[e_1^3] + [e_2^3]\}$. $w^3(X_j)$ is zero unless j=1.

The results for $j = -1,\infty$ are proved similarly, there being only one 2-cell and one 3-cell and in X_∞, $\partial[e^3]=0$.

It remains to show how the matrices $\underline{A}(k)$, $\underline{B}(n)$ are realised. For this we regard them as determining an automorphism of the second homology group, rather than an isomorphism between two copies of it. We describe briefly the general diffeomorphism obtained by Wall in [22], referring to that paper for details. Let T denote $S^2 \times S^2$ or $P \# Q$ with generators x, y of $H_2(T)$ taken respectively as b, a and $p, p+q$, i.e. such that $x.y = 1$, $y.y = 0$, and $x.x = 0.1$ respectively. Then if N is a 1-connected 4-manifold and $\beta \epsilon H_2(N)$ with $\beta.\beta = 0$ there is a diffeomorphism of $N \# T$ on itself inducing the following automorphism of $H_2(N \# T)$:

$$\begin{aligned} \gamma \in H_2(N) &\longrightarrow \gamma - (\gamma.\beta)y \\ x &\longrightarrow x + \beta \\ y &\longrightarrow y \end{aligned}$$

To produce this note that $\# T$ is equivalent to a spherical modification in N of some circle $f(S^1 \times 0)$ isotopic to zero. It is shown in [22] that β, being spherical, is also carried by an isotopy of $f(S^1 \times 0)$ finishing in its original position. By the isotopy extension theorem this is covered by an isotopy of N, the final position giving a diffeomorphism $h : N \longrightarrow N$ taking, after some adjustment, a tubular neighborhood of $f(S^1 \times 0)$ to itself. The required diffeomorphism E, of $N \# T = N - f(S^1 \times D^3) \cup_{id}$

$D^2 \times S^2$ is given by h on $N - f(S^1 \times D^3)$ and by the identity on $D^2 \times S^2$. Clearly y carried by $0 \times S^2$ is mapped to itself while x, carried in $N \# T$ by $D^2 \times 0$ together with an isotopy of $f(S^1 \times 1)$ $(1 \in \partial D^3)$ to zero in $N - f(S^1 \times D^3)$, maps to $x + \beta$. Now $\gamma \longrightarrow \gamma + ay$ since the identity is induced on $H_2(N)$ and $0 = x.\gamma = E(x).E(\gamma)$ shows that $a = -\beta.\gamma$.

For $N = T = S^2 \times S^2$ take $\beta = -a_1$ then

$$\begin{aligned} a_1 &\longrightarrow a_1 \\ b_1 &\longrightarrow b_1 + a_2 \\ a_2 &\longrightarrow a_2 \\ b_2 &\longrightarrow b_2 - a_1 \end{aligned}$$

i.e. this diffeomorphism realises $\underline{A}(1)$. $\underline{A}(k)$ is realised by repeating it k times, or by taking $\beta = -ka_1$.

Then $p_1 \longrightarrow p_1 - (p_1.(p_1 + q_1))(p_2 + q_2) = p_1 - p_2 - q_2$, and similarly $q_1 \longrightarrow q_1 + p_2 + q_2$, $p_2 \longrightarrow p_2 + \beta = p_2 + p_1 + q_1$ and, since $p_2 + q_2 \longrightarrow p_2 + q_2$, $q_2 \longrightarrow q_2 - p_1 - q_1$. This is the automorphism defined by $\underline{B}(1)$, that for $\underline{B}(n)$ is obtained by repetition, or by using $\beta = np_1 + nq_1$.

We also note here that, since there are diffeomorphisms of $P_1 \# Q_1 \# P_2 \# Q_2$ interchanging P_1 and P_2 or Q_1 and Q_2, or reversing the sign of the generator of the 2nd homology group of any one of the factors, several variants of $\underline{A}(k)$ and $\underline{B}(n)$ may be realised at once. For example $\underline{C}(n) = \begin{pmatrix} 1 & +n & +n & 0 \\ n & +1 & 0 & -n \\ -n & 0 & +1 & n \\ 0 & +n & +n & 1 \end{pmatrix}$ is obtained from $\underline{B}(n)$ by

exchanging the second and fourth rows and the second and fourth columns and then changing the signs in the new second row and the second column.

$\underline{D}(n) = \begin{pmatrix} 1 & n & n & 0 \\ 0 & n & n & 1 \\ -n & 0 & 1 & n \\ -n & -1 & 0 & n \end{pmatrix}$ is obtained by changing the

signs in the fourth column and exchanging this with the second. The manifold $X[C(n)]$, formed as X_j was but using $\underline{C}(n)$ instead of $\underline{B}(n)$, has second homology group $Z_{2n} + Z_{2n}$. Similarly $H_2(X[D(n)]) = Z_{2n-1}+Z_{4n-2}$.

We conclude this paragraph with the definition of classes $\mathfrak{M}_j$, $-1 \leq j \leq \infty$ of closed, simply connected, C^{∞}-, 5-manifolds:-

<u>Definition 1.3</u> If M is not cobordant to zero then M is in $\mathfrak{M}_{-1}$, if M is cobordant to zero and $i(w^2(M))=j$ then M is in $\mathfrak{M}_j$ (for the definition of $i(w)$ see lemma C). For M in $\mathfrak{M}_o$, define M to be in $\mathfrak{M}_{o,k}$ if, and only if, $H_2(M)=Z_k+C$ for some group C.

Clearly $\mathfrak{M}_{o,k}$ and $\mathfrak{M}_{o,m}$ are not disjoint, $M_k \# M_m$ for example being in both, but by 0.5 the classes $\mathfrak{M}_j$ are disjoint. Lemmas E and 1.1 and the above definition imply

<u>Corollary</u> 1.4 X_j is in $\mathfrak{M}_j$ and M_k in $\mathfrak{M}_{o,k}$.

2. THE THEOREMS

Theorem 2.0 (Markov [8]) The class C of closed orientable C^{∞} n-manifolds ($n > 4$) is not classifiable under diffeomorphism, combinatorial equivalence, homeomorphism or homotopy type.

Here each element of C is given by a triangulation, and by a classification would be understood the finding of a class of pairwise inequivalent manifolds, and a finite algorithm to determine, from its triangulation, to which canonical manifold an arbitary manifold of C is equivalent. The theorem follows from

Theorem 2.1 There is no algorithm for C to determine whether an arbitrary member M is simply connected.

Outline proof (for details see [8]). Given any group $G(r,k)$ with r generators and k relations between them, construct the manifold $M(r,k) =$ $D^n + h_1^1 + \dots h_r^1 + h_1^2 + \dots + h_k^2$ where the attaching maps for the 2-handles wind around the 1-handles according to the k relations. Then such an algorithm applied to $M(r,k) \cup_{id} M(r,k)$ would lead to an algorithm to determine whether $G(r,1)=1$. Adyan ([1]) has shown that such an algorithm cannot exist.

Theorem 2.2. (Smale [16]) If M is in $\mathfrak{M}_0$ then

$$M = M_{k_1} \# M_{k_2} \# \dots \# M_{k_r} \# M_{\infty} \# \dots \# M_{\infty}$$

where k_i divides k_{i+1} and $k_1 \neq 1$ unless $M = S^5$.

Proof This follows from the Poincaré theorem, proved by Smale in [14] and by Stallings and Zeeman [17] and [27] from $\Theta_5 = 0$ (Milnor [9]) and

Theorem 2.2' If M is in $\mathfrak{M}_{o,k}$ then $M = M_k \# M'$ with M' in $\mathfrak{M}_o$.

We extend Smale's results to include all simply connected 5-manifolds by proving

Theorem 2.3. If M is in $\mathfrak{M}_j$, $j \neq 0$, then $M = X_j \# M'$ for some M' in $\mathfrak{M}_o$.

Proofs of theorems 2.2' and 2.3 will be given in paragraphs 3,4 and 5.

Theorem 2.4. The class of simply connected, closed, C^∞-, 5-manifolds is classifiable under diffeomorphism. A canonical set is $X_j \# M_{k_1} \# \ldots \# M_{k_s}$ where $-1 \leq j \leq \infty$, $s \geq 0, 1 < k_1$ and k_1 divides k_{i+1} or $k_{i+1} = \infty$. A complete set of invariants is provided by the second homology group and $i(M) = i(w^2(M))$.

Proof For the definition of $i(M)$ and the proof that it is a diffeomorphism invariant see lemma C and its corollary. That this with $H_2(M)$ distinguishes between the canonical manifolds follows from lemma 1.1 and the restrictions on the k_i. For two of these manifolds can only have the same second homology group if X_j in one is replaced by M_{2j} in the other ($j>0$), in which case $i(M)$ is j for the first and zero for the second. That an arbitrary manifold of the class is diffeomorphic to one of the canonical manifolds follows from theorems 2.2 and 2.3. To discover which let $M \in \mathfrak{M}_j$ be determined by the 'extra Z_2' in $H_2(M)$ for $j=-1$ (by lemma E) and by $j=i(M)$ otherwise, and let G be left after factoring from $H_2(M)$ $Z_2, 0, Z_{2j}+Z_{2j}$, Z respectively when $j=-1, j=0$, $0<j<\infty$, $j=\infty$. Then $G=B+B+Z...+Z$ for some finite $B=Z_{k_1}+...+Z_{k_r}$, where $k_i | k_{i+1}$, and the canonical manifold which is diffeomorphic to M has factors X_j, M_{k_i}, $i=1,...r$, and one M_∞ for each Z.

There is no difficulty in describing an algorithm to determine $H_2(M)$ and $i(M)$ from the triangulation. For $i(M)$, $w^2(M)$ must be calculated on a set of generators of $H_2(M)$; this could be done using the interpretation of w^2 as the obstruction to triviality of the normal bundle of such an element, or, using lemma E, by calculating the linking numbers.

<u>Corollary 2.4.1</u> The same classification is valid for homotopy type, for combinatorial equivalence or for homeomorphism.

<u>Proof</u> The invariants, being obtainable from the homology and duality, are homotopy type invariants. Conversely diffeomorphism implies each of the above relations.

(That 1-connected 5-manifolds of the same homotopy type are diffeomorphic was shown by Novikov).

<u>Corollary 2.4.2</u> J. Cerf has shown that $\Gamma^4 = 0$ and so every combinatorial 5-manifold has a compatible differential structure and the above classification applies equally to closed 1-connected combinatorial 5-manifolds.

<u>Corollary 2.4.3</u> For each b-basis $(z_1, z_2; x_1, y_1, \ldots x_r, y_r; e_1, \ldots e_s)$ of $H_2(M,Z)$ there is a decomposition $M = M_{z_1,z_2} \# M_{x_1,y_1} \# \ldots \# M_{e_s}$, where $H_2(M_{u,v}) \cong gp\,(u,v)$ by a b-preserving isomorphism, while $H_2(Me_i) = Z$ and $w^2(Me_i) = 0 \Longleftrightarrow w^2(e_i) = 0$.

<u>Proof</u> Recall (lemma D and remark) that if the z_i are not zero then z_2 is of odd order φ, z_1 of order 2φ and $b(z_1, z_1) = 1/2$, $b(z_1, z_2) = 1/\varphi$, $b(z_i, u) = 0$ for $u = x$ or y. The pairs x_i, y_i are of order θ_i and $b(x_i, x_j) = (1/\theta_i)\delta_{ij}$ and $b(y_i, y_j) = 0$ if $i \neq j$.
The e_i are of infinite order.

<u>Construction</u>

$Me_i = M_\infty$ if $w^2(e_i) = 0$, $Me_i = X_\infty$ if $w^2(e_i) \neq 0$.

$M_{x_i, y_i} = M_{\theta_i}$ when $b(y_i, y_i)=0$, $M_{x_i, y_i} = X[C(\theta_i/2)]$ if $b(y_i, y_i) \neq 0$.

$M_{z_1, z_2} = X[D((\varphi-1)/2)]$ if $\varphi \neq 1$, $M_{z_1, z_2} = X_{-1}$ if $\varphi=1$ (and so $z_2=0$).

These manifolds have the correct second homology group and second Stiefel-Whitney class. To obtain a basis with the required linking numbers take first a b-basis which is also a U-basis as in lemma D then for $\varphi \neq 1$, or for composite θ_i, recombine as in the remark following lemma D.

Now the connected sum N of the manifolds constructed has the same homology as M. There is also a 1-1 correspondence between basis elements of their second homology groups which preserves w^2 (since on the finite elements it preserves b). Thus $i(w^2(M)) = i(w^2(N))$ and so by theorem 2.4 $M \cong N$.

Corollary 2.4.4 Corresponding to any b-basis of $H_2(M)$, M has a handle decomposition with one 0-handle and one 5-handle, and a two handle and a 3-handle for each generator of the basis. In particular it has a decomposition with the minimum number of handles consistent with its homology.

Proof This follows from the preceding corollary since each factor manifold has a decomposition with such a set of handles, this decomposition being clear from its manner of construction. $\#$ is equivalent to removing a 5-handle and a 0-handle from the disjoint sum, so the result follows for M.

To obtain a minimal decomposition, take first a U-basis which is also a b-basis and, if $H_2(M) = B + B + Z_2$ or $B + B$ and $B = Z_{k_1} + Z_{k_2} + \dots + Z_{k_r}$ with $k_i \mid k_{i+1}$, recombine to give a b-basis with $\theta_i = k_i$. Now if the extra Z_2 occurs and there are any θ_i left of odd order replace the generators z of Z_2 and x_i, y_i by $x_i + z, y_i$. This gives a basis with the minimum number of generators which is also a b-basis. The corresponding decomposition has the minimum possible number of handles.

Lemma 2.5 A simply-connected 5-manifold M immerses in R^8 ($M \propto R^8$)

(i) $M \propto R^7 \iff W^3(M) = 0$

(ii) $M \propto R^6 \iff w^2(M) = 0$

Proof (see Hirsch [5]) The only possibly non-zero Stiefel-Whitney classes are w^2 and W^3. Hence if ν is the normal bundle of the embedding of M in R^{11}, the total Stiefel class (mod 2) is $w(\nu) = 1 + w^2 + w^3$. Moreover $W^3(\nu) = W^3(M)$ since each is $\delta^*(w^2)$.

If $w^2 = o$, ν has a 5-frame cross-section over the 2-skeleton. The obstructions to extending this over M

which correspond to non-zero $H^r(M)$ have coefficients in $\pi_2(V_{6,5})$ and $\pi_4(V_{6,5})$ which are both zero, so that there is a 5-frame field over all M. Thus by [5] it is possible to immerse M in 5 fewer dimensions, that is in R^6.
Similarly if $W^3 = 0$, ν has a 4-frame section over M ($\pi_4(V_{6,4}) = 0$), and there is always a 3-frame section of ν.
Thus if $W^3(M) = 0$ $M \propto\!\subset R^7$, and in any case $M \propto\!\subset R^8$.
Conversely $M \propto\!\subset R^7$ means that, multiplying by R^4, $M \propto\!\subset R^{11}$ with a 4-frame field in the normal bundle, and so $W^3(\nu) = W^3(M) = 0$. Similarly $M \propto\!\subset R^6$ implies $w^2(M) = 0$.

<u>Corollary 2.5.1</u> $M \in \mathfrak{M}_o \implies M \propto\!\subset R^6$

$M \in \mathfrak{M}_\infty \implies M \propto\!\subset R^7$

$M \in \mathfrak{M}_j$, for any $j \implies M \propto\!\subset R^8$

and these are best possible.

<u>Proof</u> by lemmas 1.2 and 2.5 and theorem 2.4, since if $M = X_j \# M'$ with $M' \in \mathfrak{M}_o$ then $w^2(M) = w^2(X_j)$ and $W^3(M) = W^3(X_j)$.

<u>Theorem 2.6</u> (i) $M \in \mathfrak{M}_o \implies M \subset R^6$

(ii) $M \in \mathfrak{M}_\infty \implies M \subset R^8$

(iii) $M \in \mathfrak{M}_j$ for any $j \implies M \subset R^9$

These are best possible.

<u>Proof</u> (i) Consider $D^6 + h^2 = V \subset R^6$, where the 2-handle

is such that $w^2(V) = 0$ i.e. V is the trivial 4-disc bundle over S^2. Then V is simply connected and, for any k, k times the generator of $H_2(V)$ is carried by a 2-sphere S_k embedded in the boundary. The closure X of the complement of V in R^6 is 2-connected and so S_k is isotopic to zero in X by theorem B. The 3-disc formed by the isotopy may intersect itself, however since both X and this disc are 2-connected it can be approximated, using Haefliger's theorem again, by an embedded 3-disc. This disc together with its normal bundle form a 3-handle on V which makes $H_2(W) = Z_k$, where $W = V + h^3$. Clearly $W \subset R^6$ and $H_r(W) = 0$ except for $r=2$ and $H_o(W) = Z$. By duality $H_r(W,\partial W) = 0$, except for $r = 0, 6$ and $H_3(W,\partial W) = Z_k$. ∂W is simply connected and from the homology sequence of $(W,\partial W)$ it can be seen that $H_2(\partial W)$ is an extension of Z_k by Z_k, so it must be $Z_k + Z_k$, for it has an element of order k and is of the form $B + B$. Thus since $w^2(W) = w^2(V) = 0$, by theorem 2.4 $\partial W = M_k$, and taking the connected sum of these for suitable k we obtain an embedding in R^6 of the general element of $\mathfrak{M}_o$.

(ii) Any tubular neighborhood of a 2-sphere in X_∞ carrying a generator of its second homology group is a non-trivial disc bundle B. When X_∞ is immersed in R^7, S^2 and any sufficiently small tubular neighborhood is embedded. Thus we have $B \subset R^7$. Now if $B_{-1} \subset R^7 \times (-1)$

and $B_1 \subset R^7 \times (1)$ are similar embeddings, then $B_{-1} \cup_{id\, \partial B_{-1}} B \times [-1,1] \cup_{id\, \partial B_1} {-B_1} \subset R^7 \times [-1,1]$ is an embedding of $X_\infty = B \cup_{id\, \partial B} {-B}$ in R^8 once it has been smoothed.

(iii) follows by Haefliger's theorem, B.

By theorem 14 in [11] if $M^5 \subset R^{5+k}$ with normal bundle ν^k then $\chi(\nu^k) = 0$, where χ denotes the Euler class. However, if $k = 3$ $\chi(\nu^k) = W^3(M)$ which shows that (iii) is best possible, by lemma 1.2, except perhaps for $\mathfrak{M}_0$ and $\mathfrak{M}_\infty$. If $k=2$, $[\chi(\nu^k)]_2 = w^2(M)$ which shows that (ii) is best possible.

3. COBORDISMS BETWEEN THE MANIFOLDS

In each of the classes $\mathfrak{M}_j$, $j\neq 0$, and $\mathfrak{M}_{o,k}$ there is a 'special' manifold X_j and M_k respectively, all of which except X_1 are indecomposable. By definition, since $\mathfrak{M}_{-1}$ is the non-trivial cobordism class, manifolds in the same class are cobordant, and, when in paragraphs 3,4 and 5 we discuss a cobordism V between simply connected 5-manifolds X and M, we have in mind for X one of the special manifolds and for M a general manifold of the same class. Referring to the homology sequences (2) of such a cobordism it will be convenient to have the

Definition 3.1 A cobordism V between simply connected 5-manifolds X and M will be said to satisfy

($\underline{H}$), if it is simply connected and i_X and i_M are epimorphisms. It satisfies

($\underline{K}$) if it satisfies (H) and i_X is an isomorphism.

$$(2)\qquad \begin{array}{l} H_3(V,X) \xrightarrow{\partial} H_2(X) \xrightarrow{i_X} H_2(V) \longrightarrow H_2(V,X) \longrightarrow 0 \\ H_3(V,M) \longrightarrow H_2(M) \xrightarrow{i_M} H_2(V) \longrightarrow H_2(V,M) \longrightarrow 0 \end{array}$$

Since V has dimension 6, Smale's theorem A provides a minimal handle decomposition and gives geometrical significance to (H) and (K). (H) implies that in a decomposition of V on $X \times I$ there need only be 3-handles, and (K) means that these handles are attached trivially to $X \times [1]$. This will be shown in the factoring lemma (5.1) where it will be deduced that when V satisfies (K), M must have a decomposition $X \# M'$ with M' in $\mathfrak{M}_0$. Thus to prove 2.2' and 2.3 it is sufficient to find a cobordism satisfying (K) between each special manifold X and any other member M of the same class. The first step, lemma 3.2, is to take any cobordism V' and remove $\pi_1(V')$ and as much as possible of $H_2(V')$ by surgery. The result, V, satisfies (H), (Corollary 3.4), but except for the class $\mathfrak{M}_{-1}$, (Corollary 3.3), does not satisfy (K). This is because $H_2(V)$ has become smaller than $H_2(X)$ so that i_X cannot be monomorphic. In paragraph 4 V will be modified, introducing into $H_2(V)$ new elements in the image of i_X in such a way that they are also in the image of i_M so that the cobordism still satisfies (H). This is repeated until i_X has zero kernel and the cobordism satisfies (K).

<u>Lemma 3.2</u> If X,M are cobordant 1-connected 5-manifolds with second Stiefel-Whitney classes either (i) both zero or (ii) both non-zero, then a 1-connected cobordism V between them may be chosen with

in case (i), $H_2(V)=0$

and in case (ii), $H_2(V)=Z_2$ and $w^2(V)\neq 0$.

<u>Proof</u> (i) By lemma F, $w^2(M_i)=0$ implies that M_i is diffeomorphic to ∂H_i for some H_i in $\mathcal{H}(6,3,k)$. H_i has

$\pi_1(H_i)=1$, $H_2(H_i)=0$ and so for any two such M_i, $V = H_1 \# H_2$ is the required cobordism between M_1 and M_2.

(ii) When $w^2(X)\neq 0$ then neither is $w^2(V)$. For $i^* w(V) = w(\tau(V)|X) = w(\tau(X) \oplus \varepsilon^1) = w(X).w(\varepsilon^1) = w(X)$. That is $i^* w^2(V) = w^2(X)$. Here w is the total Stiefel-Whitney class and the trivial factor ε^1 is the inward normal to V along X.

On account of dimensions, e.g. by theorem B, any map of a 1-sphere or a 2-sphere into V may be approximated by an embedding; since V is orientable any embedded 1-sphere has trivial normal bundle. Surgery ([10],[7],[24]) can therefore be applied to V to obtain a simply-connected cobordism. As remarked in 0.4, if S^2 carries $x \in H_2(V)$ then $w^2(x)$ is the obstruction to triviality of the normal bundle of S^2. Thus all generators of $H_2(V)$ may be surgered out except those on which w^2 is non-zero. By 0.5 this need only be one, say x. Since $w^2(2x) = 2.w^2(x) = 0$, 2x may also be killed leaving a new cobordism V' with $H_2(V') = Z_2$.

<u>Corollary 3.3.</u> If M^5 is simply-connected and not cobordant to zero, i.e. M is in $\mathfrak{M}_{-1}$, then there is between X_{-1} and M a cobordism which satisfies (K).

<u>Proof</u> That X_{-1} is cobordant to M follows from lemma E, as noted in 1.4. It could also been seen directly from

the construction of X_{-1} given in 1 and the proof of lemma 1.2. In the notation of that lemma $\delta^{*}\{[e^2]_2\} = \{[e^3]\}$, the generator of $H^3(X_{-1})$. But $\{[e^2]_2\} = w^2(X_{-1})$ and so $w^3(X_{-1}) = \{[e^3]_2\}$. Clearly (e.g. by duality with coefficients Z_2) $w^2.w^3 \neq 0$. (Compare [3]).

Now take the cobordism V of lemma 3.2. The relation $i^{*}w^2(V) = w^2(X)$ may be written as a commutative diagram:

$$\begin{tikzcd} H_2(V) \arrow[r, "w^2(V)"] & Z_2 \\ H_2(X_{-1}) \arrow[u, dashed, "i_X"] \arrow[ur, dashed, "w^2(X_{-1})"'] & \end{tikzcd} \tag{3}$$

Since $w^2(X_{-1}) \neq 0$ and $H_2(X_{-1}) \cong H_2(V) = Z_2$, i_X must be an isomorphism. Similarly as $w^2(M) \neq 0$, i_M must be epimorphic (i_X amd i_M are as in diagram (2)).

<u>Corollary 3.4</u> The cobordism V of lemma 3.2 satisfies (H).

<u>Proof</u> In case (i) this is immediate and in case (ii) follows from the commutative diagram (3) as above.

4. REDUCTION OF THE GENERAL CASE

The cobordism V obtained in lemma 3.2 between canonical and general members, X,M respectively, of the same class $\mathfrak{M}_j$ or $\mathfrak{M}_{o,k}$ has not in general a sufficiently large second homology group for it to satisfy the hypotheses (K), which we wish to apply to complete the proof of theorems 2.2' and 2.3. An element x in $H_2(X)$ which maps to zero in $H_2(V)$ is the image, under ∂, of an element a in $H_3(V,X)$ which is carried by a disc in V whose boundary, in X, carries x. Thus to obtain a cobordism in which x maps non zero it is sufficient to do a spherical modification to a 3-sphere in V intersecting this disc in just one point, and missing discs whose boundaries carry the elements which must remain in the kernel of i_X (see diag. (2)). In particular for i_M to remain an epimorphism it is necessary that an element m in $H_2(M)$, corresponding to x, should also map under inclusion to the newly created element in $H_2(V)$. In the cobordism $W(V)$ of $X \# M$ to zero, derived from V, (see below) this is the same as asking that $x-m$ be in the kernel of the homomorphism $H_2(\partial W) \longrightarrow H_2(W)$. To find such a 3-sphere we show that a basis of $H_3(W,\partial W)$ may be chosen with some $a \in \partial^{-1}(x)$, of infinite order, as one member, and such that the image, under ∂, of the group generated by the remaining members contains just those elements which must continue to map to zero in $H_2(W)$. Then any sphere carrying v, in $H_3(W)$, dual to a with respect to this basis is suitable for the modification.

Such a modification is necessary for each generator of $H_2(X)$, and five distinct cases have to be considered. However these only differ in a few details and the proofs will be carried out simultaneously.

From now on our cobordisms V between X and M will satisfy the hypothesis (H); $\pi_1(V) = 1$ and i_X and i_M (diag. 2) are both epimorphic. This implies that $H_2(V,X) = 0 = H_2(V,M)$.

Given any cobordism V between X and M , a <u>cobordism W(V)</u> of X # M to zero may be obtained from it; join points P in X and Q in M by an arc α in V not meeting X or M again. Then α has a trivial normal bundle $\alpha \times D^5$ and W(V) may be chosen as $V - (\alpha \times \mathring{D}^5)$. Conversely V(W) can be regained from W. W(V) will be written W if no confusion can occur. Note that $H_k(\partial W) = H_k(X) + H_k(M)$ for each k, $\neq 0,6$, and for $p \leq 3$, $H_p(W) = H_p(V)$ and $H_p(W,X) = H_p(V,X)$. If V satisfies (H) $\pi_1(W) = 1$ and $i: H_2(\partial W) \longrightarrow H_2(W)$ is epimorphic.

Lemma 4.1 If V satisfies (H) then elements of $H_3(W)=H_3(V)$ are carried by embedded spheres and those of $H_3(W,\partial W)$ by embedded discs.

Proof Since $\pi_1(V) = 1$, by G.W. Whitehead's extension of the Hurewicz theorem [29], the Hurewicz homomorphism $\pi_3(V) \longrightarrow H_3(V)$ is epimorphic, i.e. elements of $H_3(V)$ are carried by maps of spheres, and by Haefliger's theorem these are homotopic to embedded spheres. From the homology sequence of $W, \partial W$, since $H_2(\partial W) \longrightarrow H_2(W) = H_2(V)$ is epimorphic, it follows that $H_2(W,\partial W) = 0$. So, by Smale's theorem, W has a handle decomposition on ∂W with no 0-, 1-, or 2-handles. Thus W has the homotopy type of ∂W with r-cells, $r \geq 3$, attached and so $\pi_2(W,\partial W) = 0$. Now by the relative Hurewicz theorem $H_3(W,\partial W) = \pi_3(W,\partial W)$ i.e. elements of $H_3(W,\partial W)$ are carried by maps of discs. Again by theorem B such maps may be approximated by embeddings, first on the boundary 2-sphere and then over the whole disc.

Lemma 4.2 If $\delta : G \longrightarrow X_1 + X_2 \dots + X_r + H$ is an epimorphism of finitely generated abelian groups with X_i cyclic and $\delta(\text{tors } G) \subset H$, and if there are at least r independent indivisible elements of G in the kernel of δ, then G may be written as a direct sum $Z_1 + \dots + Z_r + G'$ where the Z_i are infinite cyclic and $\delta Z_i = X_i$ and $\delta G' = H$.

Proof For each i let x_i be a generator of X_i, let z_1', ... z_r' be independent indivisible elements of G such that $\delta z_i' = 0$, and let $G = gp(z_1', \ldots z_r') + A$. Then $x_i = \delta a_i$ for some a_i in A. If $z_i = z_i' + a_i$ then the z_i are independent set of indivisible, and a fortiori pure, elements of G and so may be extended to a basis $z_1, \ldots z_r, y_1', \ldots y_s', t_1, \ldots t_n$, where the y_j' are of infinite order and the t_k are torsion generators. The z_i, being indivisible, are necessarily of infinite order. Let $\delta y_j' = \Sigma\lambda_{ji}x_i + h_j$ where h_j is in H and put $y_j = y_j' - \Sigma\lambda_{ji}z_i$. Then $z_1, \ldots z_r, y_1, \ldots y_s, t_1, \ldots t_n$ form a basis of G. They clearly span G since the z_i, y_j', t_k, did so it is sufficient to show they are independent. If $\Sigma\alpha_i z_i + \Sigma\beta_j y_j + \Sigma\gamma_k t_k = 0$ then $\Sigma(\alpha_i - \Sigma\beta_j\gamma_{ji})z_i + \Sigma\beta_j y_j' + \Sigma\gamma_k t_k = 0$, and since z_i, y_j', t_k are independent $(\alpha_i - \Sigma\beta_j\gamma_{ji})z_i = 0$, $\beta_j y_j' = 0$ and $\gamma_k t_k = 0$. Since y_j' and z_i are of infinite order this implies that $\beta_j = 0$ for all j and so $\alpha_i = 0$ for each i. Now $\delta y_j = h_j$ is in H, and δt_k is in H by hypothesis, so if $G' = gp(y_j, t_k)$ and $Z_i = gp(z_i)$, $\delta Z_i = X_i$ and $\delta G' \subset H$. But δ is an epimorphism so $\delta G' = H$.

This lemma will be applied to obtain special bases for $H_3(W, \partial W)$ where $W(V)$ is the cobordism associated with a cobordism V satisfying (H). In some cases $H_2(W)$

will be finite cyclic and so $H_3(W,\partial W) = H^3(W)$ will have a finite cyclic summand, and it will be necessary to know its image under $\partial : H_3(W,\partial W) \longrightarrow H_2(\partial W)$. The following description was pointed out to me by C.T.C. Wall.

<u>Lemma 4.3</u> If W is a 6-manifold with $H_2(W) = Z_k$ generated by η, there is a homomorphism $k: H_2(W) \longrightarrow Q/Z$; $\eta \longrightarrow 1/k$. The homomorphism $k \circ i$: $H_2(\partial W) \xrightarrow{i} H_2(W) \xrightarrow{k} Q/Z$ determines an element $\widetilde{\lambda}$ of $\dfrac{H_2(\partial W) \otimes Q/Z}{H_2(\partial W) \otimes Q}$, and $\widetilde{\lambda}$ and the non-singular linking form b determine an element λ of tors $(H_2(\partial W))$. Then a generator μ of tors $(H_3(W,\partial W))$ may be chosen such that $\partial\mu = \lambda$.

<u>Proof</u> Consider the commutative diagram (4).(c.f. 0.1 and diag.(1))

$$
(4) \qquad
\begin{tikzcd}
 & H_2(W)\otimes Q \arrow[rr] \arrow[dl, dashed, "\cong"'] \arrow[dd, dashed, "\alpha"] & & H_2(\partial W)\otimes Q \arrow[dl, dashed, "\cong"'] \arrow[dd, dashed, "\beta"] & \\
H_4(W,\partial W;Q) \arrow[rr] \arrow[dd, dashed] & & H_3(\partial W;Q) \arrow[dd, dashed] & & \\
 & H_2(W)\otimes Q/Z \arrow[rr, "i^*"] \arrow[dl, dashed, "\cong"'] & & H_2(\partial W)\otimes Q/Z \arrow[dl, dashed, "\cong"'] & \\
H_4(W,\partial W;Q/Z) \arrow[rr] \arrow[d, dashed] & & H_3(\partial W;Q/Z) \arrow[d, dashed] & & \\
H_3(W,\partial W;Z) \arrow[rr, "\partial"] \arrow[d, dashed] & & H_2(\partial W,Z) \arrow[rr, "i"] \arrow[d, dashed] & & H_2(W,Z) \\
H_3(W,\partial W;Q) \arrow[rr] & & H_2(\partial W,Q) & &
\end{tikzcd}
$$

Here the cokernels of α and β are isomorphic with the torsion subgroups of $H_3(W,\partial W;Z)$ and of $H_2(\partial W,Z)$ respectively, canonical isomorphisms being provided by the relevant linking forms. However $H_2(W) \pitchfork Q = 0$ and so coker $(\alpha) = H_2(W) \pitchfork Q/Z$ and the homomorphism k clearly generates it. If we define the generator μ to be the dual (by the linking form) of k, then it follows from the diagram that $\partial\mu$ and $\frac{i^*(k)}{\mathrm{im}\beta}$ are duals. However $i^*(k) / \mathrm{im}\beta$ is the element $\tilde{\lambda}$ defined in the statement of the lemma, so $\partial\mu = \lambda$.

Remark It follows from diag.(4) and a similar argument that if W is any 6-manifold with non-empty boundary, then $b(\partial x,y) = b(x,iy)$ where x,y are torsion elements of $H_3(W,\partial W)$ and $H_2(\partial W)$ respectively and b denotes the relevant linking number.

Lemma 4.4 Let X and M be simply-connected 5-manifolds and $W(V)$ be the cobordism of $X \# M$ to zero associated with a cobordism V between X and M satisfying (H), that is $\pi_1(V) = 1$ and the homomorphisms i_X, i_M of second homology induced by inclusions are epimorphic. Let $H_2(M) = H_2(X) + H$ for some group H such that $i_M(H) = 0$ and that this decomposition is orthogonal under b if $H_2(X)$ is finite. Consider the cases

(1) $H_2(W) = 0, \quad H_2(X) = Z_k + Z_k$

(2) $H_2(W) = Z_2, \quad H_2(X) = Z_k + Z_k$

(3) $H_2(W) = Z_k, \quad H_2(X) = Z_k + Z_k$

(4) $H_2(W) = 0, \quad H_2(X) = Z$

(5) $H_2(W) = Z_2, \quad H_2(X) = Z$, where k is a prime power,

then there is a spherical class v in $H_3(W)$ such that if surgery is applied to a 3-sphere carrying v, giving a modified cobordism U,

(i) if W satisfies the hypotheses for cases (1) or (2) then U satisfies those for (3), and

(ii) if W satisfies the hypotheses for cases (3), (4) or (5) then the cobordism V(U) between X and M, derived from U, satisfies (K).

Corollary 4.4.1 If X and M are simply connected 5-manifolds and V a cobordism between them satisfying (H) and such that $H_2(M) = H_2(X) + H$ for some group H with $i_M(H) = 0$ and that this decomposition is orthogonal under b if $H_2(X)$ is finite. Then if $H_2(V)$ and $H_2(X)$ satisfy cases (1), (2), (4) or (5) above (replacing W by V), there is a cobordism between X and M which satisfies (K).

<u>Proof of the lemma</u> First choose generators of $H_2(X)$ and $H_2(M)$ as follows. (We shall not distinguish between elements of $H_2(X)$ or $H_2(M)$ and their images in $H_2(\partial W) = H_2(X) + H_2(M)$.) In cases (4) and (5) let x_1 be a generator of $H_2(X)$ and m_1 of the corresponding summand of $H_2(M)$. Note that in case (5) these both map to the non-zero element of $H_2(W)$. For convenience of notation add zero 'generators' x_2 and m_2, respectively.
In cases (1), (2) and (3) let x_1', x_2' be a basis of $H_2(X)$ such that $b(x_1',x_2') = 1/k$ (see lemma D). In case (1) take $x_i = x_i'$, $i = 1,2$. In case (2) if only one of x_i' maps to the generator of $H_2(W)$ take this as x_1 and the other as x_2, if both map to the generator take $x_1 = x_1'$ and $x_2 = x_1' + x_2'$. In case (3) choose a generator u of $H_2(W)$ and let $x_1' \longrightarrow \lambda u, x_2' \longrightarrow \mu u$. Then since k is a prime power we may assume $(\lambda,k) = 1$, and can find α,β such that $\alpha\lambda + \beta k = 1$. Then $x_1 = \alpha x_1'$ and $x_2 = x_2' - \alpha\mu x_1'$ are independent generators of $H_2(X)$. $b(x_1,x_2)$ is of order k, and, choosing a suitable multiple of x_2, its value may be taken as $1/k$. Similar generators m_1,m_2 may be chosen for $H_2(M)$, using the same generator u of $H_2(W)$ in case (3), such that $H_2(M) = gp(m_1,m_2) + H$ and $b(m_i,h) = 0$ for any h in H.

Summarising, there are in all cases generators

x_1, x_2 of $H_2(X)$ and m_1, m_2 of the corresponding summand of $H_2(M)$ such that x_2 and m_2 map to zero and if $H_2(W)$ is non-trivial x_1 and m_1 map to the same generator $b(x_1, x_2) = b(m_1, m_2) = 1/k$. Note that $b(x_2, x_2) = 0$, for otherwise, by 0.4 and lemma E, a sphere carrying x_2 would have a non-trivial normal bundle in X and so also, since it is stable, in W. But this is impossible since in W it is isotopic to zero.

Now it is clear by lemma 4.3 that the image of the torsion subgroup of $H_3(W, \partial W)$ is generated in case (3) by $x_2 + m_2$ and in case (2) by $k/2.(x_2 + m_2)$. In the latter case the hypotheses imply that k is a power of 2. In case (5) it follows from elementary considerations or from the remark after lemma 4.3 that, since the homomorphism $H_2(\partial W) \longrightarrow H_2(W)$ is zero on torsion, the image of the torsion subgroup of $H_3(W, \partial W)$ in $H_2(\partial W)$ is zero.

Taking the connected sum of W with copies of $S^3 \times S^3$, the kernel of $\partial : H_3(W, \partial W) \longrightarrow H_2(\partial W)$ can be given as many infinite cyclic summands of $H_3(W, \partial W)$ as are necessary to apply 4.2.

Writing $T = gp(x_2 - m_2)$, then kernel $i : H_2(\partial W) \longrightarrow H_2(W)$ is $X + T + H$ where $X = gp(x_1,\ x_1 - m_1,\ x_2)$ in cases (1) and

(4). ($x_2 = 0$ in the latter)

$$X = gp(2x_1,\ x_1 - m_1,\ x_2) \text{ in cases (2) and (5).}$$
$$X = gp(x_1 - m_1,\ x_2) \text{ in case (3).}$$

Now writing $x = x_1$ in cases (1) and (4), $2x_1$ in cases (2) and (5), and x_2 in case (3), and $X = gp(x) + X'$,† then by lemma 4.2 $H_3(W,\partial W) = Z + F + S$ where Z is infinite cyclic, F is free and $\partial Z = gp(x)$, $\partial F = X' + H$ and $\partial S = T$. Moreover Z could be chosen with generator z such that $\partial z = x$. We are now ready to modify W; x is the element to be removed from kernel $H_2(\partial W) \longrightarrow H_2(W)$, and H, T and X' contain those elements which must remain in it.

Let v in $H_3(W)$ be dual to z for some basis of $H_3(W,\partial W)$ corresponding to this decomposition. By lemma 4.1 v is carried by a 3-sphere S^3, and since $\pi_2(SO_4) = 0$ it has a trivial normal bundle and so may be used for a spherical modification. We may assume that this normal bundle $S^3 \times D^3$ misses ∂W, and if $Y = W - [S^3 \times (\text{int } D^3)]$ then the modified cobordism is $U = Y \cup_{\text{id}S^3 \times S^2} D^4 \times S^2$. Clearly Y, and so also U, is simply connected if W is. By 4.1 elements of $H_3(W,\partial W)$ are carried by discs, and a disc D carrying an element of $F+S$ has zero intersection number with S^3. However since W is simply-connected and D and S^3 have codimension 3 the algebraic intersection

† X' is generated by $x_1 - m_1$ in case (3) and by that and x_2 in the remaining cases.

number may be realised geometrically using Whitney's method for the removal of pairs of intersections, ([26]). Thus in the modified cobordism U, which has the same boundary as W, the group X' + T + H is still in the image of ∂ and so in the kernel of $i:H_2(U) \longrightarrow H_2(U)$. A disc in U whose boundary† carries $x_1 - m_1$, gives in the associated two-ended cobordism V(U) a cylinder† $S^2 \times I$ whose ends carry x_1 and $-m_1$, so that x_1 and m_1 map to the same element in $H_2(V)$. Similarly x_2 and m_2 map to the same element. On the other hand a disc in U whose boundary carries an element of $H_2(X)$, or of $H_2(M)$, remains in V(U). Thus H is in kernel $H_2(M) \longrightarrow H_2(V(U))$ and it only remains to calculate $H_2(U)$ and the images of x_1 and x_2. Except in case (3) the image of x_2 is zero. By the excision and homotopy axioms for homology $H_k(W,S^3) \cong H_k(W,S^3 \times D^3) \cong H_k(Y,S^3 \times S^2)$ which is isomorphic similarly to $H_k(U,S^2)$. From the homology sequence of W,S^3 we see that $H_2(W) = H_2(W,S^3)$, and so from the sequence of U,S^2 we obtain the exact sequence (5).

$$(5) \qquad H_2(S^2) \xrightarrow{i} H_2(U) \longrightarrow H_2(W) \longrightarrow 0$$

Lemma 4.5 If surgery is applied to the sphere S^3 described above, and η is a generator of $H_2(S^2)$ in the

† The boundary may be taken as $S_x \# S_m$ with $S_x \subset X$, $S_m \subset M$ and $\#$ along $\alpha \times \partial D^5$ (c.f. p. 46).

corresponding sequence (5), then $i\eta$ has order k in cases (1) and (3), $k/2$ in case (2) and is of infinite order in cases (4) and (5).

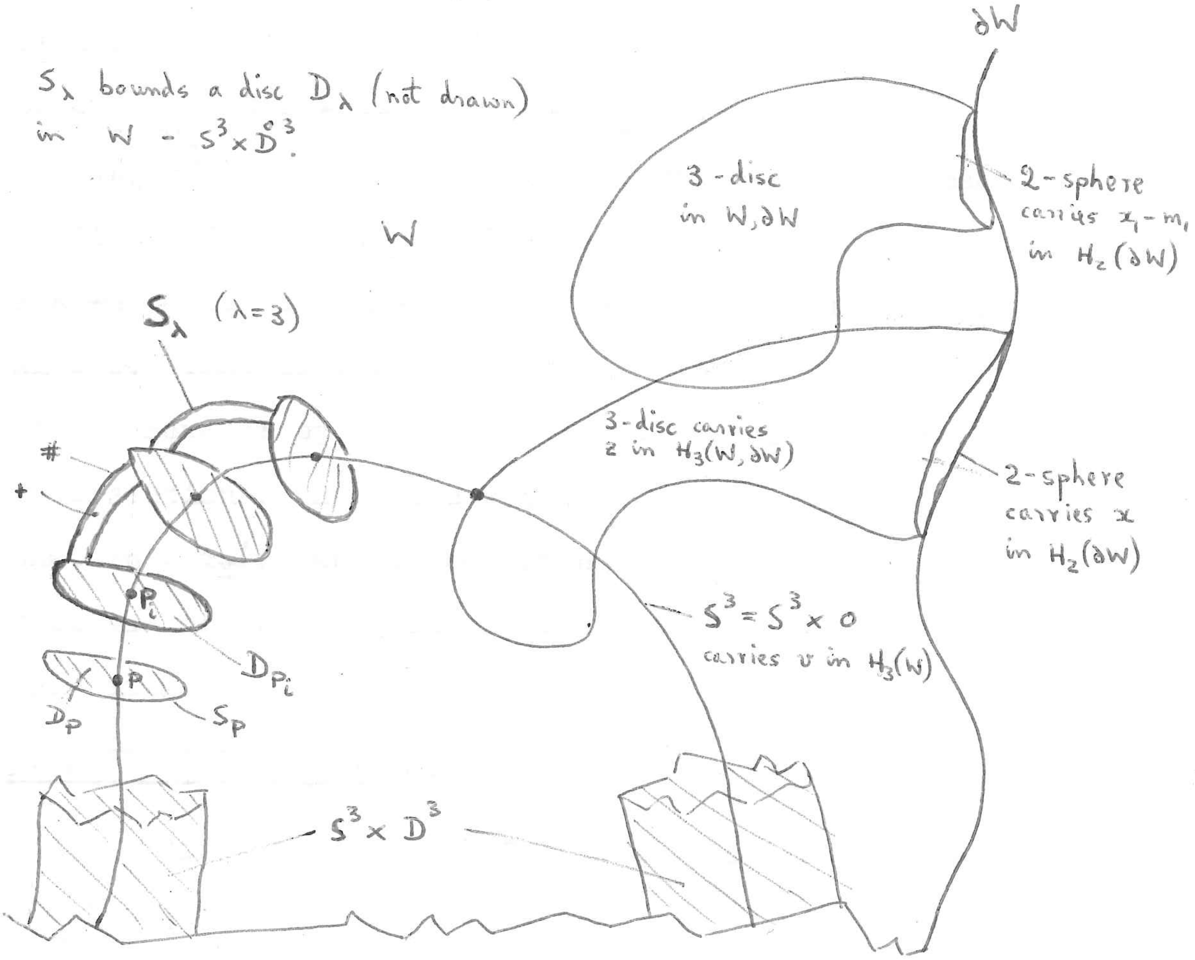


Diagram (6)

Proof See diagram (6). Let $S^3 \times D^3$ be a tubular neighborhood of S^3. Choose points P, $P_1, P_2, \ldots$ in S^3 and denote $P_i \times D^3$ by DP_i, $P_i \times \partial D^3$ by SP_i. Then η is carried by S_P both in W and in U, and if $\lambda\eta = 0$ in $H_2(U)$, $S_\lambda = SP_1 \# SP_2 \# \ldots \# SP_\lambda$ is isotopic to zero in U, the isotopy being approximable by an embedded disc D_λ in $W \cap U$. Now S_λ may be chosen as $\partial(DP_1 + DP_2 + \dots + DP_\lambda)$, the $\#$ between the spheres corresponding to the boundary of the $+$ between the discs, and then $D_\lambda \cup_{idS_\lambda} (DP_1 + DP_2 + \dots + DP_\lambda)$ is a sphere in W meeting S^3 λ times, all intersections having the same sign, so it carries an element $\lambda z+a$ where a is in $F+S$ and $\partial(\lambda z+a) = 0$. But ∂a is in $X' + T + M$ and $\partial \lambda z = \lambda x$ so $\lambda x = 0$. That is the order of $i\eta$ is a multiple of that of x. However by the choice of v, S^3 has just one intersection with a disc whose boundary carries x so that $i\eta = i_X(x)$ and $i\eta$ cannot have greater order than x. Thus the order of $i\eta$ is precisely that of x, as stated in the lemma.

Proof of 4.4. continued

Cases (1) and (4) follow at once from 4.5 and the exact sequence (5), since $H_2(W) = 0$. $H_2(U)$ is generated by $i\eta = i_X(x) = i_X(x_1)$ as required.

In cases (2) and (5) $i\eta$, $= i_X(x) = i_X(2x_1) = 2.i_X(x_1)$ is divisible by two, and by no more since otherwise (5) could not be exact. Thus in each case $H_2(U)$ is generated by $1/2.i\eta = i_X(x_1)$, and its order is the same as that of x_1 in $H_2(X)$, i.e. k, ∞ respectively.

In case (3), there is by lemma 4.5 an exact sequence

(7) $$0 \longrightarrow Z_k \longrightarrow H_2(U) \longrightarrow Z_k \longrightarrow 0$$

A generator of the second Z_k, which is $H_2(W)$, is $i_X(x_1)$. An element u in the kernel of the homomorphism $H_2(\partial U) \longrightarrow H_2(U)$ is carried by the boundary of a 3-disc in U, and since W is obtained from U by a spherical modification on a 2-sphere this disc is also in W. Thus u also maps to zero in $H_2(W)$, i.e. the kernel of $H_2(\partial U) \longrightarrow H_2(U)$ is a subgroup of that of $H_2(\partial W) \longrightarrow H_2(W)$, $(\partial W = \partial U)$. Since therefore the orders of the images of x_1 in $H_2(W)$ and in $H_2(U)$ are the same we have a homomorphism of $H_2(W)$ back to $H_2(U)$. Moreover since this and the homomorphisms in (7) are induced by inclusions it is clear that it is a splitting homomorphism for the sequence. Thus $H_2(U) = Z_k + Z_k$ with one summand generated by $i_X(x_1)$ and the other by $i\eta = i_X(x_2)$. This completes the proof of lemma 4.4.

5. THE FACTORING LEMMA AND PROOF OF THEOREMS 2.2' AND 2.3.

Lemma 5.1 Let X, M be simply-connected 5-manifolds which are cobordant by a simply-connected 6-manifold V satisfying the hypotheses (K) i.e. such that in the sequences

$$(2)\qquad \begin{array}{l} H_3(V,X) \xrightarrow{\partial} H_2(X) \xrightarrow{i_X} H_2(V) \longrightarrow H_2(V,X) \longrightarrow 0 \\ H_3(V,M) \longrightarrow H_2(M) \xrightarrow{i_M} H_2(V) \longrightarrow H_2(V,M) \longrightarrow 0 \end{array}$$

i_X is an isomorphism and i_M an epimorphism, then $M = X \# M'$ for some M' in $\mathfrak{M}_o$.

Proof By exactness $H_2(V,X) = 0 = H_2(V,M)$ and already $H_1(V,X) = H_1(V,M) = 0$, so by duality and the universal coefficient theorem $H_4(V,X) = 0$, and $H_5(V,X) = 0$. Clearly $H_o\ (V,X) = H_6(V,X) = 0$.

Now since V is 6-dimensional and V, X and M are 1-connected, Smale's theorem A applies to give a decomposition of V on $X \times I$ with only 3-handles. These handles carry generators of $H_3(V,X)$, but i_X is monomorphic and so by exactness $\partial = 0$ and each handle attaching map is homologous to zero in $X \times [1]$ (if $I=[0,1]$). Theorem B then implies that these attaching maps are isotopic to zero. That is that each may be smoothly deformed to a position inside a 5-disc D^5 in $X \times [1]$. With care these isotopies may be performed simultaneously, and we outline a method for doing this. If the attaching maps are $f_i : S_i^2 \times D^3 \longrightarrow X \times [1]$, $i=1,\dots k$, note that it is sufficient to deform the embedding $f_i | S_i \times [0]$, for then by the isotopy extension theorem this may be extended to an isotopy of the tubular neighborhood and, once $S_i \times [0]$ is inside D^5, $S_i \times D^3$ may be pulled inside by the tubular neighborhood theorem. The case $k = 3$ with $S_1 \subset D^5$ will illustrate the procedure. If $F = S_2 \times I$ is (the image of) an isotopy of S_2 to a position in D^5 we may assume it only meets S_1 and S_3 in isolated points. Using the methods of [26] and [28] these may be 'pushed along arcs' over the boundary $S_2 \times [1]$ of F; they may not be pushed across $S_2 \times [0]$ since this would alter the linking numbers of the original embeddings. To keep S_1 in D^5 the arc chosen for a point P of $S_1 \cap F$ must be in D^5. If this is not already possible and $P = (x,t)$ in F, join $P' = (x,t')$, where $t' > t$ and $x \times [t,t']$ is in D^5, to a point of $S_2 \times [1]$ by arcs α in $D^5 - F$ and β in F. Since $\pi_1(X) = 1$ α and β form the boundary of a disc D^2 missing $S_1, S_2 \times [0]$, and S_3 and only meeting F along β (pushing other intersections across $S_2 \times [1]$). F may then be 'moved across' D^2 (see [26]) to obtain a new isotopy which contains α and has no more intersections with S_1 or S_2 than F did.

When all the attaching maps are inside D^5, V is in the form $V = (X \times I) + (D^6 + h_1^3 + \ldots + h_k^3)$ where the first $+$ takes place in the boundary component $X \times [1]$. Thus V has connected components diffeomorphic to X and $X \# M'$, where $M' = \partial(D^6 + h_1^3 + \ldots + h_k^3)$ and so, by lemma F, M' is in $\mathfrak{M}_0$. Since these components are known to be diffeomorphic to X and M this completes the proof of the lemma.

<u>Proof of theorems 2.2' and 2.3.</u>

We have to show that if X is a 'special' manifold, i.e. one of those constructed in paragraph 1, and M is a general manifold in the same class as X, then $M = X \# M'$ where M' is in $\mathfrak{M}_0$. By lemma 5.1 it is sufficient to find a cobordism between X and M which satisfies (K). This has already been achieved for the class $\mathfrak{M}_1$ by corollary 3.3, so we consider $\mathfrak{M}_j$, $j > 0$, and $\mathfrak{M}_{o,k}$, $k > 1$. It is clearly also sufficient to consider only the values of k which are powers of primes.

We shall show that the cobordism V between X and M obtained in lemma 3.2 satisfies all the conditions for one of the cases of lemma 4.4 and so by corollary 4.4.1 deduce the existence of a cobordism satisfying (K).

By corollary 3.4 V satisfies (H). If M is in $\mathfrak{M}_{o,k}$, $1 < k < \infty$, then $H_2(M)$ has by definition a factor Z_k and by lemma D it has a second such factor. (The extra Z_2 occurs only in the non-trivial cobordism class.) Similarly if M is in $\mathfrak{M}_j$, $H_2(M)$ has a factor $Z_{2}j$, a generator of which is non-zero under w^2, and again it must also have another. When M is in $\mathfrak{M}_{o,\infty}$ or $\mathfrak{M}_{\infty}$ $H_2(M)$ has a factor Z. Thus by lemma D(and the remark following it) there is in each case a decomposition $H_2(M) = G + H$ which is orthogonal for the linking form b, when G is finite and such that $G = H_2(X)$, this isomorphism preserving b. The condition that $i_M(H) = 0$ is redundant when M is in $\mathfrak{M}_{o,k}$, for then the cobordism obtained in 3.2 has $H_2(V) = 0$. When M is in $\mathfrak{M}_j$, $o < j < \infty$, corresponding cobordism has $H_2(V) = Z_2$, and from the diagram (3) it can be seen that the elements of $H_2(M)$ which map to the non-zero element are those on which w^2 is non-zero, i.e. those x such that $b(x,x)=1/2$ (see lemma E).

Now using the complement to lemma D, we can arrange that there is only one such generator in the decomposition $G + H$, and moreover that this is one of the generators of G. When M is in $\mathfrak{M}_{\infty}$, $G=Z$ and as in lemma C we may arrange (by varying the decomposition $H_2(M) = G+H$) that its generator is the only one non-zero under w^2. Thus in all cases we can satisfy the additional condition $i_M(H) = 0$.

Applying case (1) of 4.4.1 when X and M are in $\mathfrak{M}_{o,k}$, $1 < k < \infty$, case (2) when they are in $\mathfrak{M}_j$, $0 < j < \infty$, case (4) when they are in $\mathfrak{M}_{o,\infty}$ and case (5) when they are in $\mathfrak{M}_{\infty}$, there is a cobordism between X and M which satisfies the hypotheses (K) and so by lemma 5.1, $M = X \# M'$ where M' is some manifold of $\mathfrak{M}_o$. This completes the proofs of the theorems.

REFERENCES

[1] S.I. Adyan, 'Algorithmic insolubility of the problem of determining certain properties of groups', Dokl.Akad.Nauk., S.S.S.R, 103 (1955 pp.533-535.

[2] J. Cerf, 'Topologie de certains espaces de plongements', Bull. Soc. Math. France, 89 (1961) pp. 227-380.

[3] A. Dold, 'Erzeugende der Thomschen Algebra $\mathfrak{N}$', Math.Zeit., 65 (1956) pp.25-35.

[4] A. Haefliger, 'Plongements différentiables de variétés dans variétés', Comment.Math.Helv., 36 (1961) pp.47-82.

[5] M.W. Hirsch, 'Immersions of manifolds', Trans. Amer.Math.Soc., 93 (1959) pp.242-276.

[6] I. Kaplansky, Infinite abelian groups, University of Michigan, 1954.

[7] M.A. Kervaire and J. Milnor, 'Groups of homotopy spheres I', Ann. of Math. 76 (1962) pp 504-537.

[8] A.A. Markov, 'The insolubility of the problem of homeomorphy', Proc.Int.Congress of Math. 1958, Cambridge 1960.

[9] J. Milnor, Differentiable manifolds which are homotopy spheres, (mimeographed), Princeton University 1959.

[10] J. Milnor, 'A procedure for killing homotopy groups of differentiable manifolds', Differential Geometry: Symposia in pure mathematics III, Amer.Math.Soc. 1961.

[11] J. Milnor, Lectures on characteristic classes, (mimeographed), Princeton University 1957.

[12] G.F. Paechter, 'On the groups $\pi_r(V_{m,n})$, I', Quart.J.Math. (Oxford) 7 (1956) pp.249-268.

[13] H. Seifert and W. Threlfall, Lehrbuch der Topologie, Leipzig 1934.

[14] S. Smale, 'Generalised Poincaré's conjecture in dimensions greater than four', Ann. of Math. 74 (1961) 391-406

[15] S. Smale, 'On the structure of manifolds', Amer.J.Math., 84 (1962) 387-399.

[16] S. Smale, 'On the structure of 5-manifolds', Ann. of Math., 75 (1962) 38-46.

[17] J.R..Stallings, 'Polyhedral homotopy spheres', Bull.Amer.Math.Soc. 66 (1960) 485-488.

[18] N.E. Steenrod, The topology of fibre bundles, Princeton 1951.

[19] R. Thom. 'Quelques propriétés globale des variétés différentiables', Comment.Math.Helv., 28 (1954) 17-86.

[20] C.T.C. Wall, 'Killing the middle homotopy groups of odd dimensional manifolds', Trans.Amer.Math. Soc., 103 (1962) 421-433.

[21] C.T.C. Wall, Differential topology seminar notes, (mimeographed), Cambridge University 1962.

[22] C.T.C. Wall, 'Diffeomorphisms of 4-manifolds', Journal London Math. Soc., 39 (1964) 131-140.

[23] C.T.C. Wall, 'Classification of (n-1)-connected (2n+1)-manifolds', to appear.

[24] A.H. Wallace, 'Modifications and cobounding manifolds, I', Canad.J.Math. 12 (1960) pp.503-528.

[25] A.H. Wallace, '-----------------, II', J.Math. and Mech. 10 (1961) 773-809

[26] H. Whitney, 'The self-intersections of a smooth n-manifold in 2n-space', Ann. of Math., 45 (1944), 220-246.

[27] E.C. Zeeman, 'The generalised Poincare conjecture', Bull.Amer.Math.Soc., 67 (1961) p.270.

[28] R. Penrose, J.H.C. Whitehead, and E.C. Zeeman, 'Imbedding of manifolds in Euclidean space', Ann.of Math., 73 (1961) 613-623.

[29] G.W. Whitehead. 'On spaces with vanishing low-dimensional homotopy groups', Proc.Nat. Acad.Sci., U.S.A., 34(1948) 207-211.

CHAPTER II

THE STRUCTURE OF MANIFOLDS

In this chapter we consider the problem of simplifying handle decompositions of manifolds, being particularly interested in the case of non-simply connected manifolds. We shall only consider the C^{∞} category although most of the techniques are easily adapted to the combinatorial case. In particular the symbol $\cong$ between manifolds will always represent a diffeomorphism. In general our manifolds will be compact. We shall omit discussion of the rounding of corners in product manifolds, assuming it to be carried out wherever necessary.

The ideas and notation follow closely those of S. Smale and, as in his paper on the structure of manifolds [11], the main theorem is one on the realisation of prescribed handle decompositions of a manifold (theorem 6.1), but we are unable to obtain the kind of canonical ('minimal') decomposition which Smale obtained in the simply-connected case. The corollaries deduced are straightforward translations of those obtained by Smale and include the s-cobordism theorem (6.3) which corresponds to his h-cobordism theorem and has also been proved by B. Mazur in [6]. There is also a duality theorem (6.9) for the torsion in an h-cobordism related to J. Milnor's results in [8].

Theorem 6.1 is proved by the manipulation of the handles and in paragraph 5 we study this for decompositions having only r- and (r+1)-handles for each r. The cases r=0 and r=1 are special and are dealt with first. For the general case there are three types of manipulation; insertion of handles which is always possible (1.2), rearrangement of the handles of each type which requires $r < n-2$ (5.8) and cancellation of certain pairs of handles which requires $r < n-3$ (5.7 and 5.9).

Our approach to the cancellation lemma differs from that of Smale in that, instead of quoting H. Whitney's result on the removal of intersections between submanifolds which are superfluous for the algebraic intersection, we measure the obstruction to applying Whitney's method ([15]). This obstruction is closely related to the simple homotopy type torsion of J. H. C. Whitehead ([13]), and indeed for an h-cobordism equals it (5.7). We express this relation geometrically by constructing certain characteristic paths of the decomposition (paragraph 2) and showing that they both measure the obstruction (2.2) and determine the structure of the covering complex in which the torsion is measured(3.1). They also measure the effect on the covering complex of the movement of handles (3.4) by which 5.8 is proved, and allow a simple description of the duality (3.6) concerned in 6.9.

In paragraph 1 we recall the definitions and main techniques of handle theory, together with an outline of the proofs. Paragraph 4 contains an account, largely without proofs, of torsion and simple homotopy type sufficient to indicate the point of view we adopt and to justify the results used in the later paragraphs.

I am grateful to C.T.C.Wall and E.C.Zeeman for pointing out errors in the original draft of this work and for suggesting improvements in the presentation.

1. HANDLE DECOMPOSITIONS.

If Mand N are n-manifolds and f an embedding of a submanifold Q of bN, the boundary of N, into bM, denote by $M + f(N)$ the manifold formed from the disjoint union of M and N by identifying Q with its image f(Q). When $N = D^r \times D^{n-r}$ and $Q = S^{r-1} \times D^{n-r}$ this adds an <u>r-handle</u> to M and the notation for the resulting manifold is abbreviated to $M + f(h^r)$ or $M + h^r$. A handle has two sets of characteristic spheres; attaching or a-spheres $f(S^{r-1} \times x)$, and belt or b-spheres $y \times S^{n-r-1}$. <u>The a-sphere</u> and <u>the b-sphere</u> will mean respectively $f(S^{r-1} \times 0)$ and $0 \times S^{n-r-1}$, where 0 denotes the centre of the relevant disc.

If N is an n-dimensional submanifold of M a <u>decomposition of M on N</u> is a diffeomorphism d of N+handles onto M where, if Q is the union of the connected components of bN to which the handles are added and $Q \times I$, $I = [0,1]$, a tubular neighborhood of $Q = Q \times 0$ in N, then d is the inclusion on $N - Q \times [0,1/2)$. If $N = Q \times I$ then d is called a decomposition of M on Q. We shall only consider finite decompositions, that is those in which the number of handles is finite, and we shall usually abuse the notation and refer also to the manifold N+handles as a decomposition of M.

The results of this paper will depend on the possibility of performing isotopies of the handle attaching maps in a decomposition, starting from a given isotopy of an a-sphere, to obtain a new decomposition in which that a-sphere is in the position it occupies at the end of the isotopy. Let $d : N + f(h^r) + \ldots \xrightarrow{\cong} M$ be a decomposition of M on N. An isotopy of the a-sphere $f(S^{r-1} \times 0)$ extends by the isotopy extension theorem to an isotopy $H : Q \times I \dashrightarrow Q \times I$. Then, if π is the projection of $Q\times I$ on Q and $H_t(x) = \pi H(x,t)$ for x in Q, $N + f(h^r)$ is diffeomorphic to $N + (H_1 \circ f)(h^r)$ by an extension $\overline{H}_1$ of H_1 which is the identity on $N - Q \times [0,1/2)$. For we may define $\overline{H}_1(q,t) = (H_{1-2t}(q),t)$ for $t \leq 1/2$, and since H_1 takes $f(S^{r-1} \times D^{n-r})$ to $H_1 \circ f\,(S^{r-1} \times D^{n-r})$, the attaching map of $H_1 \circ f(h^r)$, it may be extended, for example by the 'identity', over h^r. If the next handle in the original decomposition is $g(h^s)$, form $N + H_1 \circ f(h^r) + \overline{H}_1 \circ g(h^s)$ and extend $\overline{H}_1$ as before over h^s. Repetition of this for all handles gives the required decomposition $d \circ \overline{H}_1^{-1} : N + H_1 \circ f(h^r) + \ldots \overset{\sim}{\dashrightarrow} M$. (Diagram 1).

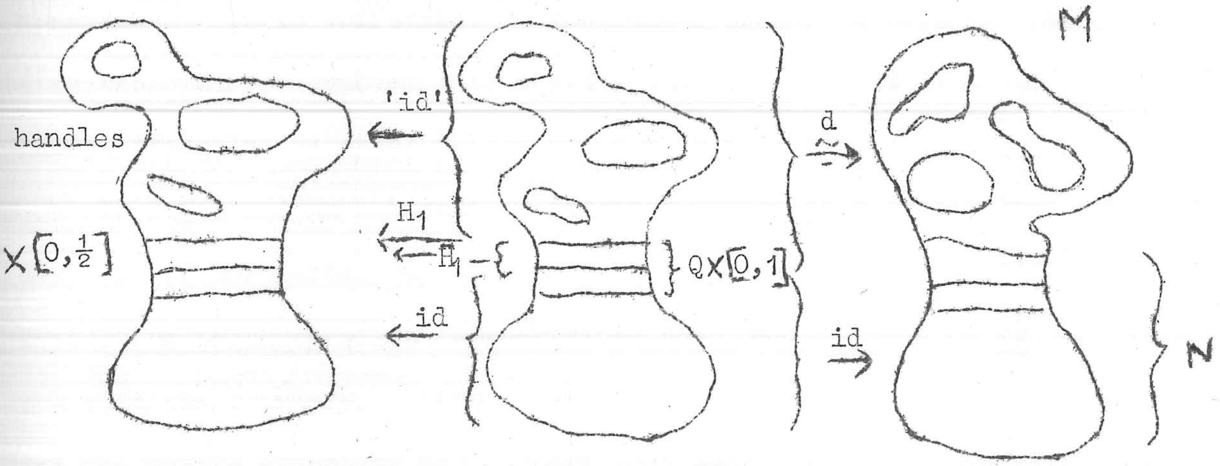


Diagram 1

Note that in this new decomposition, although the intersection of all the a-spheres with bN has been changed, provided $\overline{H}_1$ is extended by the 'identity' over each handle their intersection with other handles, and in particular with b-spheres, is not changed. Similar results may be obtained when the a-sphere we wish to move does not belong to the first handle, simply replacing N by N + the previous handles in the above discussion.

If λ is a critical value of a differentiable function $f : M \dashrightarrow R$ such that $[\lambda - \delta, \lambda + \delta]$ has only one critical point, and that has index λ, then a λ-handle is really $f^{-1}[\lambda - \delta, \lambda + \delta]$ together with the trajectories of f; the intersections of the unstable and stable manifolds with the boundary being, respectively, the a-sphere and b-sphere. Thus we should have defined a handle as $Q \times I + f(h^r)$ and it is for this reason that $\overline{H}_1$ above cannot be constant on all of N. $Q \times [0,1/2]$ is really an essential part of the handle. However the given definition is more suitable for manipulating the handles, for which attention is focussed on the expressions N+handles. It is sufficient to know that, as the handles are rearranged, a suitable diffeomorphism onto M still exists. This definition and the manipulating lemmas which follow are due to S. Smale [10].

If $r \leqslant s$ and an s-handle is attached before an r-handle then the a-sphere of the latter may be moved, by an isotopy, to a position where it avoids the b-sphere of the former since this has codimension greater than r - 1 in the boundary. Then it may be pushed 'radially' off the handle and the entire attaching map 'contracted' until it is sufficiently close to its a-sphere for it too to miss the s-handle. After this it is clearly possible to add the r-handle first and, by repeating the procedure a finite number of times, to arrange that all the handles are added in ascending order of their index r, with handles of the same index having disjoint attaching maps. We shall usually assume this normalisation carried out on any decomposition we wish to consider. So to visualise a decomposition

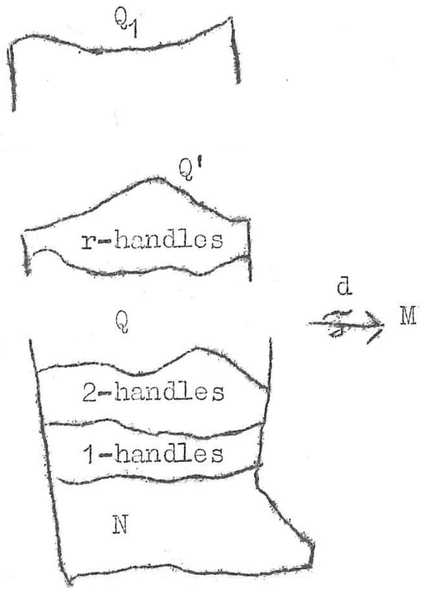


Diagram 2.

imagine a height function on M which puts the (r+1)-handles above the r-handles for each r (diag. 2). Thus if the handles are attached to the same component of bN we refer to the corresponding component Q_1 of bM, obtained after adding all the handles, as the 'top' boundary.

A normalised handle decomposition corresponds to a function constant on N, non-degenerate on $\overline{M-N}$ and such that the value at a critical point is the index. The existence of such functions when $\overline{M-N}$ is compact was first shown by Smale in [9].

Associated with a decomposition of M on N there is a decomposition of $\overline{M-N}$ on Q_1, derived as follows. Let an r-handle be attached to Q, and Q' be the new boundary (diag.1). Then the handle $D^r \times D^{n-r}$ might equally be considered as an (n-r)-handle attached by $D^r \times S^{n-r-1}$ to Q'. With all the handles reinterpreted in this manner we obtain the <u>dual</u> decomposition of $\overline{M-N}$ on Q_1; as Q_1 is identified by d with the corresponding boundary of M, we may use the restriction of d for the associated diffeomorphism.

With the handle decomposition of M on N we associate a <u>cell decomposition</u> of K on N, where K is a cell complex rel N ($N \subset K$ and K - N is the union of disjoint open cells, such that the closure of each r-cell is attached by its boundary to $K^{(r-1)}$, the union of N and the cells of dimension less than r in K - N). We shall call K the <u>complex associated with the handle decomposition</u> of M on N. To obtain K the handles are put in normal order and each collapsed onto the cell 'at its centre', working from the top of the decomposition downwards. We shall only use the complex associated with a decomposition which has just r- and (r+1)-handles, and this will be described in more detail.

First collapse the (r+1)-handles onto their central discs, and then write each r-handle $D^r \times D^{n-r}$ as
$D^r \times S^{n-r-1} \times I / \{(x,y_1,0) \sim (x,y_2,0)\}$ for arbitrary x in D^r and y_i in S^{n-r-1} ($I = [0,1]$), and writing U for the union in $S^{r-1} \times S^{n-r-1} \times [1]$ of its intersections with the images under attaching maps of the a-spheres $S^r \times [0]$ of all the (r+1)-handles, collapse
$D^r \times S^{n-r-1} \times I$ onto $(U \times I) \cup (D^r \times S^{n-r-1} \times [0])$ modulo the above identifications. After the collapse a 'membrane' remains under each component of U. Since the a-spheres are attached by embeddings, such a component V of U is a submanifold of an a-sphere, which is the boundary of the cell e^{r+1} which remains after collapsing the corresponding (r+1)-handle. Thus the membrane forms a 'chimney' on the cell e^{r+1} and, e.g. by the 'chimney lemma', when added to it still forms a cell. (The chimney lemma states that if Q is a submanifold of bM then $M + Q \times I$ with corners rounded is diffeomorphic to M). The identifications determine the extent to and manner in which e^{r+1} is attached to the new cell $e^r = D^r \times . \times [0]$. The diagram 3 indicates what happens for 1- and 2- handles. When a handle is considered as $Q \times I + h^r$ the cell e^r is extended by the membrane formed by the projection through $Q \times I$ of the intersection of the boundary of e^r with $Q = Q \times 0$.

Taking a triangulation compatible with the differential structure of M and with the handle decomposition of M on N it can be seen that this collapsing may be performed as a sequence of 'elementary collapses' in the sense of Whitehead i.e. in his notation $M = DK$ rel N, and so it follows from theorem 13 of [13] that K has the same simple homotopy type rel N as M ($K \equiv M$ rel N (Σ)). This construction clearly generalises to the case when there are handles of any number of types.

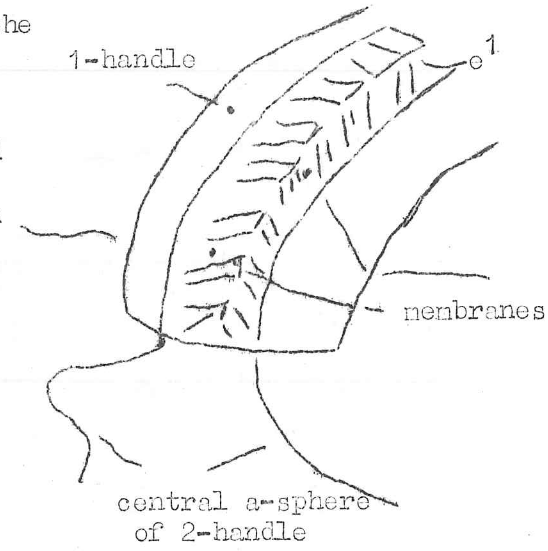


Diagram 3.

An r-handle and an (r+1)-handle are <u>complementary</u> if the a-sphere of the latter intersects the b-sphere of the former transversely and only once.

LEMMA 1.1 If h^r, h^{r+1} are a complementary pair of handles then $M + h^r + h^{r+1}$ is diffeomorphic to M.

PROOF. Collapse the (r+1)-handle onto that part of the image of its attaching map which does not meet the r-handle, pulling the latter with it: if the a-sphere of the (r+1)-handle meets the b-sphere $0 \times S^{n-r-1}$ transversely in the one point $0 \times y$, then it meets similarly the spheres $x \times S^{n-r-1}$ for say $|x| \leq 4\delta$ and, after an obvious isotopy, we may assume it meets them in (x,y) for $|x| \leq 3\delta$. The disc $(|x| \leq 3\delta) \times y$ then has two tubular neighborhoods, one the natural one N in the boundary of the r-handle and the other the image of its neighborhood in the (r+1)-handle, the latter being twisted with respect to the first. After sufficiently contracting the whole tubular neighborhood of the a-sphere we may assume that the fibres over x for $|x| \leq 2\delta$ all lie within N. Then we may perform an isotopy which leaves the fibres over x for $|x| > 2\delta$ fixed and moves the fibres over x for $|x| \leq \delta$ so that they first lie in the corresponding fibre of N and then that they coincide with it, the isotopy being extended over the remaining fibres by means of a 'bump function'. After a further isotopy, 'expanding' the disc $x \leq \delta$ and 'pushing' the twisted fibres off the r-handle, h^r and h^{r+1} meet along an (n-1)-disc in their boundaries and so together form an n-disc. This meets M along a manifold of the form $S^{r-1} \times D^{n-r} + D^r \times D^{n-r-1}$ where D^{n-r-1} is in bD^{n-r}, S^{r-1} is bD^r and the identification is the trivial one along $S^{r-1} \times D^{n-r-1}$. Thus the n-disc meets M along an (n-1)-disc and we have a manifold diffeomorphic to M.

COROLLARY 1.2. Complementary handles of any index may be inserted into a decomposition.

PROOF. $D^{n-1} \cong S^{r-1} \times D^{n-r} + id(D^r \times (S^{n-r-1} - D_o^{n-r-1}))$ where D_o^{n-r-1} is a small disc in $S^{n-r-1} = bD^{n-r}$. Thus if x is in $S^{n-r-1} - D_o^{n-r-1}$ $S^{r-1} \times x$ bounds the disc $D^r \times x$ in D^{n-1}. Let D^{n-1} be a disc in the boundary Q of any stage M of the decomposition. Then if we attach an r-handle, $B^r \times D^{n-r}$, by the identity over $S^{r-1} \times D^{n-r}$ we may attach an (r+1)-handle over $D^r \times x + id(B^r \times x)$ since this necessarily has trivial normal bundle. As these handles are clearly complementary the result M_1 is diffeomorphic to M, and indeed the diffeomorphism may be taken as the inclusion outside a tubular neighborhood of D^{n-1}. The resulting diffeomorphism between Q and the new boundary may be used to add the remaining handles as it was above when performing isotopies of the attaching maps, and in this way we shall arrive at a new decomposition which includes the extra complementary pair of handles.

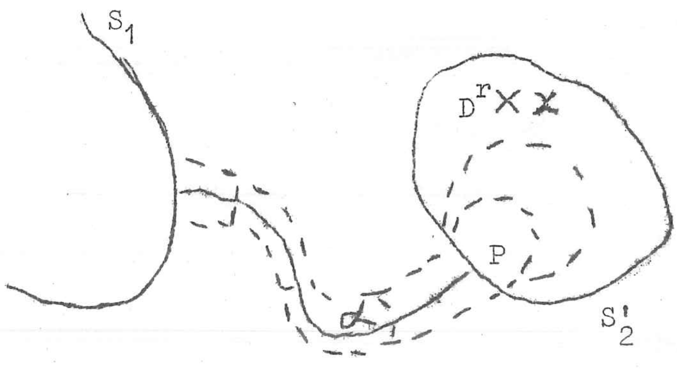
NOTE that by a suitable choice of the disc D^{n-1} we may assume that the a-spheres of the complementary pair avoid b-spheres of previous handles, and that their b-spheres avoid the a-spheres of later handles.

COROLLARY 1.3. If a complementary pair occur in a normalised decomposition then they may be removed.

PROOF. Add the r-handle concerned in the pair after the remaining r-handles and the (r+1)-handle next. They may then be cancelled by 1.1 and the resulting diffeomorphism of the boundaries used as before to add the remaining handles.

LEMMA 1.4 If r-handles are added to M^n with a-spheres S_1^{r-1} and S_2^{r-1} then the same result may be obtained with handles having a-spheres $S_1 \# (\pm S_2')$ and S_2 provided $n - r > 1$, where S_2' is isotopic to S_2 in bM and either sign + or - is possible.

PROOF By the argument given above it is sufficient to add the second handle and exhibit, in the boundary of the result, an isotopy of S_1 with $S_1 \# (\pm S_2')$. If the second handle is $D^r \times D^{n-r}$ where S_2 is $bD^r \times [0]$ then for S_2' we may take $bD^r \times x$, for some x in bD^{n-r}. To obtain the isotopy deform S_1 along an arc α joining S_1 to S_2', so that S_1 now



Stages in the isotopy of S_1 are represented by dotted lines.
Diagram 4.

meets S'_2 in a disc P. Since S'_2 bounds the disc $D^r \times x$ in the boundary of the second handle, P may be deformed isotopically across this disc onto $S'_2 - \overset{\circ}{P}$. In this position the sphere represents $S_1 \# \pm S'_2$ for some choice of sign determined by the isotopy along α. Since the codimension, $(n-1) - (r-1)$, of the spheres in bM is at least 2 this isotopy may be rechosen to give the other sign.

REMARK The effect of this on the associated complex K is to replace the cells e_1, e_2 which correspond to the handles by cells f_1, f_2 such that $bf_1 = be_1 \# \pm be_2$, $bf_2 = be_2$.

LEMMA 1.5 If an r-handle is added over the component Q of bM, where $\pi_k(M,Q)=0$ for $k \leq s$ and $0 < r < n-s-1$, then $\pi_k(M+h^r, Q')=0$ for $k \leq s$ where Q' is the boundary component corresponding to Q.

PROOF $h^r = D^r \times D^{n-r}$. The effect on Q is to remove $S^{r-1} \times D^{n-r}$ and to add $D^r \times x$ and $D^r \times (S^{n-r-1} - x)$, for some x in $S^{n-r-1} = bD^{n-r}$. The latter is an (n-1)-disc and has no effect on π_k if $n-1 > k+1$, and the removal of $S^{r-1} \times D^{n-r}$ has no effect if $(r-1)+(k+1) < n-1$. Thus if $0 < r < n-s-1$ only $D^r \times x$ can affect $\pi_k(M,Q)$ for any $k \leq s$. But any element of $\pi_k(M+h^r, Q')$ can be represented by a map of a k-disc, differentiable on the inverse image of $M-(Q \times [0,1/2))$ and transverse on say $Q \times 1$, where $Q \times [0,1]$ is a tubular neighborhood of $Q = Q \times 0$ in M. Then the inverse image of $(M-(Q \times [0,1)), Q \times 1)$ forms a C.W. pair of dimensions (k,k-1) which may, since the relative homotopy groups of $(M-(Q \times [0,1)), Q \times 1)$ up to the kth vanish, be deformed into $Q \times 1$ leaving fixed anything which is already there. This provides a homotopy of the original k-disc into Q'.

The following will be referred to as <u>handle moves</u>:-

(1) Insertion or removal of a complementary pair of handles

(2) Isotopy of handle attaching maps. (Including those which just modify the tubular neighborhood of the a-sphere, or those which also move the a-sphere, for example as in lemma 1.4.)

Decompositions of M on N will be called <u>equivalent</u> if they differ by any sequence of handle moves. Strictly a handle move is the diffeomorphism between equivalent decompositions but it is easier to visualise the manipulation of the handles which produces it.

For further details of the proof of 1.1 and 1.4 see [10] and [11], for a discussion of the rounding of corners and associated problems see [1], [3] and [12], and for a detailed discussion of the relation between a decomposition and its associated complex which leads to an alternative proof of the s-cobordism theorem see Mazur's paper [6].

2. THE CHARACTERISTIC PATHS

Let M^n have a decomposition on N in which only r- and (r+1)-handles occur for some r, $1 < r < n-1$, and let these handles be attached to the same component Q of bN. (The construction which follows can be carried out when r = 1 but does not then have the same significance).

Using the obvious isotopies (compare the proof of 1.1), ensure that if the a-sphere of an (r+1)-handle meets a b-sphere of an r-handle then it 'goes right round' that handle, i.e. that it meets all the b-spheres of that handle in the same number of distinct points. Now choose some base point $*$ in Q which does not lie in the image of any handle attaching map.

CONSTRUCTION 2.1 It is possible to find a disc D^* in Q, containing the base point $*$ and such that for each s-handle $D^s \times D^{n-s}$, s = r and s = r + 1, there is a point P^*_a in S^{s-1} such that $P^*_a \times D^{n-s}$ is contained in D^*.

NOTE In the following we shall not distinguish between S^{s-1} and $S^{s-1} \times [0]$, referring to both as S_a the a-sphere of the s-handle, and we shall now take $P^*_a \times S^{n-s-1}$ for the b-sphere S_b, choosing some point P^*_b in the latter. Neither do we distinguish between points of $S^{s-1} \times D^{n-s}$ and their images under the attaching map. We refer to D^* as the base disc and to P^*_a and P^*_b as base points or subsidiary base points of their spheres.

PROOF First choose a disc D_o in Q avoiding all handles, and choose points P^*_a such that $P^*_a \times D^{n-s}$ is contained in Q, misses D_o, and avoids other handles. To achieve this it may be necessary first to 'thin' the handles, that is regarding D^{n-s} as the unit disc to replace D^{n-s} by λD^{n-s} for some $\lambda < 1$, and to modify the r-handles as in lemma 1.1.

Choose further a disc D_a about P_a^* in S_a such that $D_a \times D^{n-s}$ also lies in Q, missing D_o and the other handles. Then $D_a \times D^{n-s}$ is an (n-1)-disc D_1 and may be joined to D_o by an isotopy α between an (n-2)-disc in bD_o and one in bD_1. This isotopy may be taken to miss D_o, D_1 and other handles since $(n-1) - r > 0$, and as it is also an (n-1)-disc the result is a new disc D_o' which contains $P_a^* \times D^{n-s}$.

This procedure may be repeated for all the handles to give a disc D^* with the required properties. (Diagram 5).

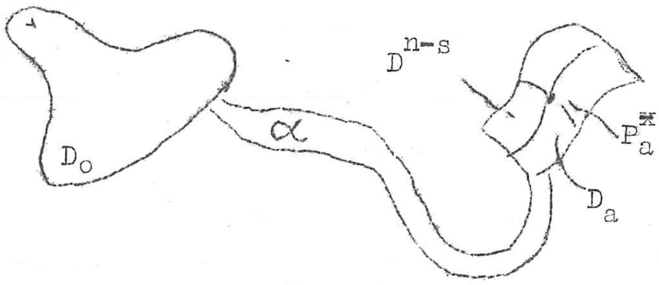


Diagram 5

Now let S_a, S_b belong to (r+1)- and r-handles respectively, with orientations chosen. To each (isolated) point P_i in the intersection of S_a with S_b we associate a sign σ_i, the sign of the intersection, and a loop π_i in Q. The latter is $* P_a^* P_i P_b^* *$ (see diagram 6), where the first and last segments are in D^* and miss handles, $P_a^* P_i$ is in S_a and $P_i P_b^*$ in S_b. We assume further that the last segment goes from P_b^* to $P_{a'}^*$, in $P_{a'}^* \times D^{n-r}$, and thence to $*$ in $S_{a'}$, the a-sphere of the r-handle (This is not essential but clears a point in 3.1). We denote by π_i also the class of the loop π_i in $\pi_1(Q,*)$.

DEFINITION The pair (π_i, σ_i) will be called the <u>characteristic</u> of the intersection P_i.

REMARK 1. π_i is not an invariant of the decomposition since clearly the segments $* P_a^*$ and $* P_b^*$ may, on varying D^*, be replaced by non-isotopic segments. What does remain invariant is the relative characteristic

$\pi_i \circ \pi_j^{-1}$, however it will be convenient to have the characteristic defined as above to provide a basis for the operations of the fundamental group and correspondingly, in the next paragraph, to index the cells of the universal cover of the associated complex.

REMARK 2. When the ambient manifold Q is not orientable it is still possible to define a relative intersection number between any P_i and P_j such that $\pi_i \circ \pi_j^{-1}$ is orientable. For then the loop $\pi_i \circ \pi_j^{-1}$ has trivial normal bundle and intersection numbers at P_i and P_j defined in it will either always be the same or always differ. Defining the relative intersection number to be +1 or -1 respectively, it is clear that this is the number which is concerned in Whitney's procedure for removing intersections. (This is needed in the proof of lemma 2.2).

LEMMA 2.2 If the intersections P_i, P_j, P_k, ... of S_a with S_b have characteristics (π_i, σ_i), ... such that $\pi_i = \pi_j$ and $\sigma_i = -\sigma_j$, and $1 \leqslant r < n-3$, then S_a, and with it the attaching map of the (r+1)-handle, may be moved isotopically to a position where it intersects S_b only in the points P_k ... with the same characteristics as before. [i.e. the decomposition has been replaced by an equivalent one in which two fewer intersections occur between a-spheres of (r+1)-handles and b-spheres of r-handles].

PROOF Without loss of generality we may assume that we are concerned with the last r-handle and the first (r+1)-handle to occur in the decomposition and look only at these two, provided we check that the characteristics of the remaining intersections are not affected. Now $\pi_i \circ \pi_j^{-1}$ is the loop

$* P^*_b P_j P^*_a * P^*_a P_i P^*_b *$, which is homotopic to zero if and only if $P^*_b P_j P^*_a P_i P^*_b$ is. But the latter is a loop through P_i and P_j formed of arcs in S_a and S_b and homotopic to similar arcs α, β tubular neighborhoods of which avoid the segments $P^*_a P_k$ and $P^*_b P_k$ for each remaining intersection P_k. (This needs $r > 1$ so that S_a is at least a 2-sphere). Let Q_1 denote that part of Q which does not meet the first r-handles, then $\pi_1(Q,*) \cong \pi_1(Q_1,*)$ since the

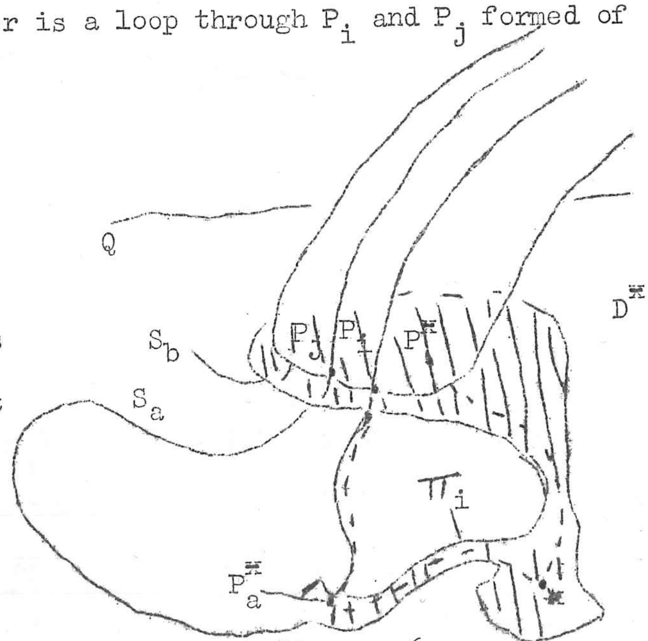


Diagram 6

parts removed from Q have essential dimension r-1 and so have codimension more than 2 in Q^{n-1}. Since α is homotopic to β in Q_1 which has dimension at least 5 and in which S_a has codimension at least 3, it is possible, using Haefliger's theorems [4], to span $\alpha \cup \beta$ by an embedded disc D^2 which avoids S_a except along α. As above the removal of $S^{r-1} \times \overset{\circ}{D}{}^{n-r}$ will not alter the fundamental group so D^2 may also avoid the r-handle except along β. Finally D^2 may miss those segments of characteristic paths which are in D^*. Since the intersections P_i and P_j have opposite signs and S_a has codimension at least 3, D^2 or more precisely a small tubular neighborhood of it, may be used in the manner of Whitney [15] to remove these intersections. As this neighborhood need not meet the characteristic paths of the remaining intersections, these paths need not be changed and so clearly their characteristics (π_k, σ_k) will not have been affected. (The idea of Whitney's procedure for removing the pair of intersections is to find a field of (r-1)-frames over D^2 in its normal bundle, which are tangent to S_a (of dimension r)

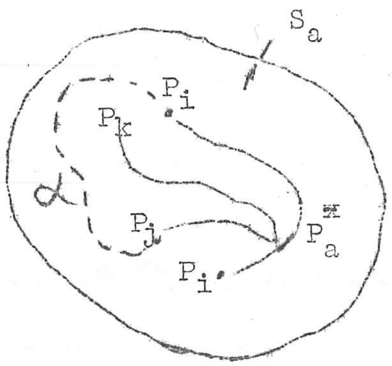


Diagram 7

along α and are normal to S_b along β. This field is then used to describe a deformation of (the embedding of) S_a 'over' that which moves α across and then off the disc past β. This may clearly be performed in any given tubular neighborhood of D^2. The details are in paragraphs 10-12 of [15]).

3. THE COVERING COMPLEX

Assume now that the inclusion of N in M induces an isomorphism of fundamental groups, where M and N are as in the previous paragraph. Let K be the associated complex rel N, and denote by e_i^s the cell of K-N which is derived from the handle h_i^s and the handle decomposition of M and N. If $P^* = P_a^*$ is the subsidiary base point of the a-sphere of this handle then P^* is in e_i^s. The universal covering space $\tilde{K}$ of K is a complex rel $\tilde{N}$, having cells (e, π) for each cell e of K-N and each element π of $\Pi = \pi_1(K) = \pi_1(N)$. By abuse of notation we refer to $\tilde{K}$ as the covering complex of K or even of M. Choose orientations for the cells of K-N and $\tilde{K}-\tilde{N}$ and choose a base point $\tilde{*}$ over $*$. Then each cell $\tilde{e}$ of $\tilde{K}-\tilde{N}$ has a point $\tilde{P}^*$ over the base point P^* of the underlying cell e of K-N. Let π be the class in Π of the loop in K which is formed by the image under the covering map of the path in $\tilde{K}$ joining $\tilde{*}$ to $\tilde{P}^*$ together with a segment P^** in D^*. Then $\tilde{e} = (\sigma e, \pi)$, where σ is +1 if the chosen orientation of e is that induced from the covering map from the chosen for e, and $\sigma = -1$ otherwise. To fix the idea diagram 8 shows the naming of a cell $\tilde{e}$ in the covering complex of a circle. The latter is formed of cells e and $*$, the disc D^* is R$*$S, and p^*, $*$ and orientations are chosen as indicated. The 'naming loop' goes round the circle, in the positive direction, past $*$ to P^* and then back in D^* to $*$. Thus it carries a generator g of the fundamental group, and $\tilde{e}$ is (-e, g).

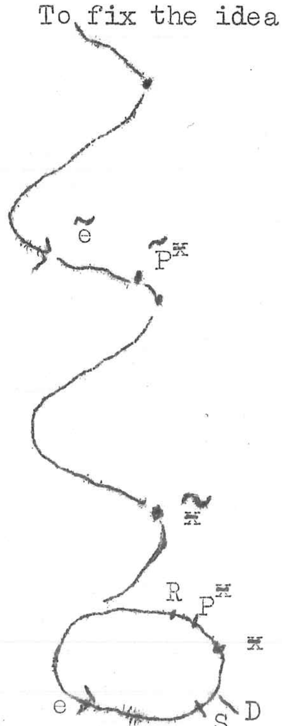


Diagram 8

The elements of Π operate on $\tilde{K}$ mapping $\tilde{N}$ to itself and permuting the cells of $\tilde{K}-\tilde{N}$. Thus $C_*(\tilde{K},\tilde{N})$ is a free $Z[\Pi]$-module, where $Z[\Pi]$ is the integral group ring of Π, and it has a preferred basis determined, up to sign and multiplication by elements of Π, by the cells of K-N. The boundary homomorphism b in $C_*(\tilde{K},\tilde{N})$ is a $Z[\Pi]$-homomorphism. It is determined by

LEMMA 3.1

$$b(e^{r+1},1) = \sum(\sigma_j e^r_k, \Pi_j)$$

where we denote a cell of $\tilde{K}-\tilde{N}$ and the corresponding basis element of C_* by the same symbol, the summation $\sum$ extends over all intersections P_j of the a-sphere S_a of the handle h^{r+1} with b-spheres of r-handles h^r_k, and (Π_j, σ_j) is the characteristic of P_j.

PROOF Since the sphere S_a either goes completely round the r-handle or misses it altogether, for each P_j there must be some $\tilde{e}^r$ in $b\tilde{e}^{r+1}$, and since the rest of the boundary is in $\tilde{N}$ the summation given is sufficient. If the a-sphere of h^{r+1} meets the b-sphere of h^r in P_j which has sign σ_j then σ_j is clearly also the sign of the corresponding cell in $b(e^{r+1},1)$. But $\tilde{e}^{r+1} = (e^{r+1},1)$ means that the path $*\tilde{P}^*_a$ projects onto a path homotopic to the path $*P^*_a$ chosen in D^*, so that we may lift the loop $\Pi_j = *P^*_a P_j P^*_b *$ to a path, in $\tilde{K}$, starting $*\tilde{P}^*_a$, continuing through $\tilde{e}^{r+1}$ to $\tilde{P}_j$, to $\tilde{P}^*_b$ and finally along a segment over $P^*_b *$, which was constructed to pass through the base base point P^*_a, of e^r. Thus Π_j is the naming loop of the cell $\tilde{e}^r$ in $b(e^{r+1},1)$ which corresponds to P_j, and so $\tilde{e}^r = (\sigma_j e^r, \Pi_j)$.

COROLLARY 3.2 If h^{r+1} is attached trivially, that is by an a-sphere which is isotopic to zero, then $b(e^{r+1},1) = 0$.

THE PROOF follows at once from the lemma since the attaching map may be moved to a position where it meets no handles and the summation above is then empty.

COROLLARY 3.3 If h^r is a member of a trivial complementary pair h^{r-1}, h^r, i.e. if h^{r-1} and h^r are attached over a disc meeting no b-spheres of previous handles nor a-spheres of later ones, then (e^r, π) does not appear in $b(e^{r+1}, 1)$ for any r+1 cell e^{r+1} or any π.

PROOF. Immediate. The result is also valid if the disc meets b-spheres.

In lemma 3.1 the boundary homomorphism b was expressed in terms of the preferred bases determined by the handles. If the handles are moved the expression for b will undergo a corresponding change, and it is the possible extent of such a change which will determine our results. In particular if it is possible to arrange that $be^{r+1} = e^r$ then from 3.1, 2.2 and 1.3 it will follow that the corresponding handles may be removed. The two following lemmas describe the changes in b which may be achieved by handle moves.

We continue to assume that K is the complex associated with a decomposition of M on N in which there are only r- and (r+1)-handles and in which the inclusion of N in M induces an isomorphism $\Pi = \pi_1(N) \cong \pi_1(M)$. Assume also that all the r-handles are added to the same component Q of bN and that the (r+1)-handles are added to the corresponding component of the boundary of the new manifold.

LEMMA 3.4 If $\pi_1(Q) \cong \pi_1(N)$ by inclusion and $1 < r < n-2$ then, if e_1, e_2 denote the base elements of $C_{r+1}(\tilde{K}, \tilde{N})$ corresponding to two of the (r+1)-handles and π is any element of Π, the first handle may be moved, giving a new base element f_1, to satisfy either of the following conditions:

(i) $b(f_1) = \pm \pi b(e_1)$

(ii) $b(f_1) = b(e_1) \pm \pi b(e_2)$

Moreover this may be achieved without changing the boundaries of the other base elements of $C_{r+1}(\tilde{K},\tilde{N})$.

PROOF (i) The element π^{-1} is carried by a circle in Q, which can be assumed to miss the a-spheres of the r-handles and so remain in the boundary after they are added. Let P^* be the subsidiary base point of the a-sphere of the first (r+1)-handle. Move P^* around the circle and back into the base disc D^*, extending to an isotopy of the attaching map in the usual way. Now the path in $\tilde{K}$ from $\tilde{*}$ to the original $\tilde{P}^*$ has to be extended by a segment over the circle carrying π^{-1} to reach the new position of $\tilde{P}^*$ and so the name of each cell over e_1^{r+1} is multiplied by π^{-1} to obtain the corresponding cell over f_1^{r+1}, where e_1^{r+1}, f_1^{r+1} are the cells corresponding to the two positions of the handle. Then if $f_1 = (f_1^{r+1}, 1)$,

$$b(f_1) = b(e_1^{r+1}, \pi) = \pi b(e_1^{r+1}, 1) = \pi b(e_1).$$

For the choice $f_1 = (-f_1^{r+1}, 1)$ we get similarly $b(f_1) = -\pi b(e_1)$.

(ii) This is achieved by the move of lemma 1.4, where the path α now goes from P_1^* in S_1 around a circle carrying π^{-1} to the base point P_2^* in S_2'. Thus we add to S_1, S_2' with its base point moved around π, and by an argument similar to the above it can be seen that $b(f_1) = b(e_1) \pm \pi b(e_2)$. It is clear that the isotopies used here could be chosen to avoid the a-spheres of other (r+1)-handles, in which case the boundaries of the corresponding base elements would not be altered.

There are two decompositions and hence two complexes concerned in the above lemma, but the natural identification between those parts which precede the handle which is moved (c.f. paragraph 1) justifies our notation. Indeed the diffeomorphism between the two equivalent decompositions induces a homotopy equivalence between the associated complexes (inclusion of the first complex followed by the diffeomorphism and then projection onto the second complex) which in turn induces an equivalence, (in the terminology of paragraph 4 a simple equivalence), between the associated based chain complexes. From the definitions it is clear that this identifies a base element corresponding to a handle which is not moved with the corresponding base element in the new decomposition. In our case we have the

ADDENDUM. If $g : C \dashrightarrow C'$ is the equivalence of based chain complexes induced by the equivalence of decompositions in 3.4, then $g(e) = e'$, where e and e' are the respective base elements corresponding to a handle which is not moved, and in case

(i) $g(e_1) = \pm \pi^{-1} f_1$,

(ii) $g(e_1) = f_1 \mp \pi e_2'$.

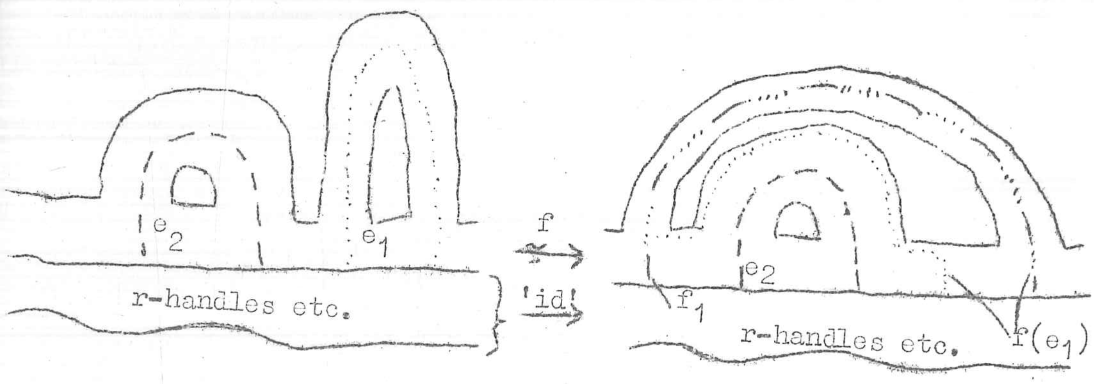


Diagram 9

PROOF. This follows at once from the preceding discussion and the expression for the boundaries in the statement of the lemma. Alternatively it can be deduced directly from the definitions of the diffeomorphism associated with the given handle move and of the associated complexes. A pictorial indication is given in diagram 9.

Let (π_k, σ_k) be the characteristic of an intersection P_k of the a-sphere of the (r+1)-handle h_i^{r+1} with the b-sphere of h_j^r and denote B_{ij} the sum, in $Z[\Pi]$, of $\sigma_k \pi_k$ over all such intersections, then 3.1 says $b(e_i^{r+1},1) = \sum_j B_{ij}(e_j^r,1)$. If the (r+1)-handles concerned in 3.4 are the ith and kth respectively then the effect on the matrix B of (i) is to multiply the ith row by $\pm\pi$ and of (ii) is to add $\pm\pi$ times the kth row to the ith row. That is, in the terminology of the next paragraph, B has in each case been premultiplied by a $[\pm\Pi]$-elementary matrix. The insertion of a complementary pair of r- and (r+1)-handles into the decomposition introduces an extra row and column into B having as 1 as their common entry and zeros elsewhere. If premultiplication by an elementary matrix of order greater than the number of rows of B is understood to include a suitable number of such extensions of B, then B may be premultiplied by any $[\pm\Pi]$-elementary matrix. That is

COROLLARY 3.5 With the data of 3.4, if B is the matrix of the boundary homomorphism in $C_*(\tilde{K},\tilde{N})$ with respect to the preferred bases, then there is an equivalent decomposition with matrix EB where E is any $[\pm\Pi]$-torsion free matrix of any order.

There is a close connection between 3.1 and the corresponding result in the dual decomposition. Let $\mathcal{D}$ be a decomposition of M on $Q \times I$, having only r and (r+1)-handles, $1 < r < n-2$, and such that $\pi_1(Q) \cong \pi_1(M) \cong \pi_1(Q')$ by inclusions, where Q is connected and Q' the corresponding 'top' component of bM, (by lemma 1.5 this restriction is superfluous unless r = 2 or n-3), and let $\mathcal{D}^*$ be the dual decomposition of M on Q'. Let e_i^s correspond to the s-handle h_i^s of $\mathcal{D}$ and let e_i^{n-s} be the cell corresponding to the dual handle h_i^{n-s} of $\mathcal{D}^*$. Let $\bar{\ }$ denote the antiautomorphism of $Z[\Pi]$ induced by replacing each element π of Π by its inverse π^{-1}.

LEMMA 3.6. With the hypotheses and notation above, if b is the boundary homomorphism in $C_*(\tilde{K},\tilde{Q})$, b^* that in $C_*(\tilde{K},\tilde{Q}')$ and

$$b(e_i^{r+1},1) = \sum_j B_{ij}(e_j^r,1) \text{ then}$$

$$b^*(e_j^{n-r},1) = \sum_i \pm \bar{B}_{ij}(e_i^{n-r-1},1).$$

PROOF The a-sphere and b-sphere of a handle of $\mathcal{D}$ correspond respectively to the b-sphere and a-sphere of the dual handle in $\mathcal{D}^*$. If π_k is the characteristic path of an intersection P_k between the a-spheres S_a of an (r+1)-handles and the b-sphere S_b of an r-handle of $\mathcal{D}$ then it is clear from the description in paragraph 2 (see diagram 6) that the base disc D'^* in Q' may be so chosen, coinciding with D^* in $Q \cap Q'$, that the characteristic path of the intersection P_k between S_a and S_b, in their dual roles in D^*, is homotopic in M to π_k^{-1} (since it now passes through P_b^* before P_a^*, i.e. its direction is reversed). Identifying $\pi_1(Q)$ with $\pi_1(Q')$ under the product of the natural isomorphisms $\pi_1(Q) \longrightarrow \pi_1(M) \longleftarrow \pi_1(Q')$, if (σ_k, π_k) is the characteristic of P_k in $\mathcal{D}$, then its characteristic in $\mathcal{D}^*$ is $(\pm\sigma_k, \pi_k^{-1})$, and the lemma is proved by 3.1. **The sign is $(-1)^{r(n-r-1)} = (-1)^{rn}$.**

4. WHITEHEAD TORSION AND SIMPLE HOMOTOPY TYPE

This paragraph contains a summary of the definitions and results we shall require and is given largely without proofs. It is based on the work in [2], [7], and [13], mainly the last, and for fuller proofs and further details reference should be made to these papers.

Let R be a ring with identity element 1 and such that every free left R-module of finite rank has unique rank. R need not be commutative. Let V be a subgroup of the multiplicative subgroup U of two-sided units. Let G_n be the group of $n \times n$ matrices over R with two-sided inverses. Then $U = G_1$ and for each i, G_i may be regarded as a subgroup of G_{i+1}, the imbedding being defined by bordering a matrix of G_i, to the right and below, diagonally with 1 and elsewhere with zeros. Similarly $G_i < G_{i+r}$ for any $r > 0$. Define <u>G(R)</u> $= \bigcup_n G_n$.

<u>A V-elementary matrix</u> is a square matrix having zeros in all positions except for either (i) a 1 in each diagonal position and one off-diagonal entry from R, or (ii) a 1 in each diagonal position except one which has an entry from V. An $n \times n$ V-elementary matrix is in G_n. Let $E_V(R)$ be the subgroups of G(R) generated by the V-elementary matrices; matrices of $E_V(R)$ will be called <u>V-simple</u>. Whitehead showed that $E_{[1]}(R)$, where $[1]$ denotes the subgroup with only the element 1, is the commutator subgroup $[G,G]$ of $G = G(R)$. Thus the <u>Whitehead group</u> $W_V(R) = G(R) \,/\, E_V(R)$ is abelian and will be written additively; it is also equal to $W_{[1]}(R) \,/\, V$. The image in $W_V(R)$ of an element a of G(R) will be called the <u>V-torsion of a</u> and written $t_V(a)$, or $t(a)$ if V is clear from the context. Since a free left R-module

has unique rank, the matrix of a change of basis is in some G_n and its V-torsion will be called the V-torsion of the change or <u>V-torsion between the bases</u>. Given a module isomorphism Θ between based free left R-modules, i.e. modules in which bases $e_1, \ldots, e_r$ and $f_1, \ldots, f_r$ respectively are given, then the <u>V-torsion of Θ</u> is defined to be that between $\Theta e_1, \ldots \Theta e_r$ and $f_1, \ldots f_r$. Clearly this is unaffected by a V-torsion-free change of basis in either module. It is the V-torsion of the matrix representing Θ with respect to the given bases.

LEMMA 4.1 If $e_1, \ldots e_k$, $f_1, \ldots f_k, \ldots f_n$ and $e_1, \ldots e_k$, $g_1, \ldots g_n$ are bases of a free left R-module then for any V there is a V-torsion-free change of the second basis to one containing $f_1, \ldots f_k$.

PROOF. It is sufficient to consider $V = [1]$, i.e. to use only those elementary matrices which have a 1 in every diagonal position. After the obvious elementary moves we may assume, using the summation convention, that

$g_j = a_{jm} f_m$ and

$f_i = b_{ij} g_j + c_{il} e_l$, $i, j, m = 1, \ldots, n$, $l = 1, \ldots k$ so $f_i = b_{ij} a_{jm} f_m + c_{il} e_l$ from which it follows that $b_{ij} a_{jm} = \delta_{im}$, and $c_{il} = 0$.

Now let $g_j \longrightarrow g'_j = g_j - a_{jm} e_m$, $e_i \longrightarrow e_i$ $\quad i, m = 1, \ldots k$ and $j = 1, \ldots n$.

Then let $e_i \longrightarrow e_i + b_{ij} g'_j$, $g'_j \longrightarrow g'_j$ $\quad i, m = 1, \ldots k$ and $j = 1, \ldots n$.

Since $e_i + b_{ij} g'_j = e_i + b_{ij}(g_j - a_{jm} e_m) = b_{ij} g_j = f_i$, and since the matrices of the changes used have 1's in all diagonal positions the lemma is proved. (The matrices are clearly products of elementary ones)

We can now define the torsion of an acyclic complex of based free R-modules as indicated by J. Milnor in [7]. (This definition will also work for quasi-free mudules). An <u>elementary complex</u> is one which is non-zero in two consecutive dimensions only with the boundary between them having zero torsion. A <u>collapsible complex</u> is a direct sum of these. Complexes are considered modulo the addition or removal of collapsible direct summands. We call bases of an R-module equivalent when they differ by changes of zero torsion including, when another based module is added, the extension of both by the same base elements, and call the equivalence classes <u>volumes</u>. Then from 4.1 it can be seen that in the short complex

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

volumes in two of A, B, C will determine one in the third. For example, given volumes in B and C, after adding the trivial elementary complex

$$0 \longrightarrow E \xrightarrow{id} E \longrightarrow 0 \longrightarrow 0,$$

choose a representative base in B some elements of which map to the given ones in C and the remaining elements of which map to zero and so lift to a base of A. That the corresponding volume is well determined follows at once since a base in C together with non-equivalent bases in A determine (the sequence splits) non-equivalent bases in B. Now given the acyclic complex

$$0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \longrightarrow C_o \xrightarrow{d} 0$$

with volumes given in each C_q use the short complexes

$$0 \longrightarrow dC_q \longrightarrow C_{q-1} \longrightarrow dC_{q-1} \longrightarrow 0$$

to determine a new volume in C_r, for some r, and define the torsion of the complex to be $(-1)^r$ times that between this volume and the given volume in C_r. Note that since a collapsible complex has zero torsion, the torsion of any acylic complex is well defined. It is consistent with the definition used by Whitehead. For the latter (see [2], [13]) fold over the complex, using the lifting homomorphism $\eta : C_o \longrightarrow C_1$ such that $d \circ \eta = \mathrm{id}C_o$ to replace

$$\ldots \longrightarrow C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_o \longrightarrow 0$$

by

$$\ldots \longrightarrow C_2' = C_2 + C_o \xrightarrow{d+\eta} C_1 \longrightarrow 0,$$

continue until we have $0 \longrightarrow C_n' \longrightarrow C_{n-1}' \longrightarrow 0$ and call $(-1)^{n-1}$ times the torsion of this the torsion of the original complex.

More generally if $f: C \longrightarrow C'$ is a chain homotopy equivalence of based complexes, define $(C(f), d_f)$ to be the acyclic complex with base elements e_n', αe_{n-1} in dimension n for each base element e_n' of C_n' and e_{n-1} of C_{n-1} and $d_f(e_n') = d'(e_n')$, $d_f(\alpha e_{n-1}) = f(e_{n-1}) - \alpha d(e_{n-1})$. Define $t(f)$ to be $t(C(f))$ and call f a <u>simple equivalence</u> when $t = 0$. Whitehead showed (Theorem 9 and lemma 3 of [13]) that $t(f) = 0$ if, and only if, there are collapsible complexes B and B', zero in any dimension $< p$ or $> q$ when C and C' are both zero in dimensions $< p$ and $> q$, and a <u>simple isomorphism</u>‡ $g : C + B \longrightarrow C' + B'$ (i.e. $g_n : C_n + B_n \longrightarrow C_n' + B_n'$ has zero torsion for each n)‡, such that f is chain homotopic to $k \circ g \circ i$, where $i : C \longrightarrow C + B$ and $k : C' + B' \longrightarrow C'$ are the natural inclusion and projection.

NOTE that in the special case when the acyclic complex C is non-zero only in dimensions r and r + 1 then $t(C)$ is $(-1)^r\, t(d)$ where d is the only non-zero differential. If $f : C \longrightarrow D$ is an equivalence between two such complexes then $C(f)$ is

$$0 \longrightarrow \alpha C_{r+1} \longrightarrow D_{r+1} + \alpha C_r \longrightarrow D_r \longrightarrow 0,$$

$\ddagger$ For the rest of this paper we shall only call an isomorphism between complexes <u>simple</u> when the matrix of each g_n is already a product of elementary matrices.

and the boundary isomorphism in D may be used for the lifting map to fold this into $0 \dashrightarrow D_r + \alpha C_{r+1} \dashrightarrow D_{r+1} + \alpha C_r \dashrightarrow 0$, giving a matrix

$$\begin{pmatrix} B_D^{-1} & F_{r+1} \\ 0 & -B_C \end{pmatrix}$$

where B_D is the matrix of the boundary in D and F_{r+1} the matrix of f_{r+1}, so that $t(f) = t(D) - t(C)$. Similarly when each of f_r and f_{r+1} is an isomorphism but C and D not necessarily acyclic, F_r may be used to lift D_r and then the boundary in the new complex has matrix $\begin{pmatrix} 0 & F_{r+1} \\ F_r^{-1} & -B_C \end{pmatrix}$ so that in this case $t(f) = t(f_r) - t(f_{r+1})$. For the corresponding calculations for general complexes see [2].

The following is a special case of lemma 3 of [7].

LEMMA 4.2 If $0 \to C' \to C \to C'' \to 0$ is an exact sequence of based acyclic complexes, each zero except in dimensions r and r+1, such that the bases of C are the sum of the image of those in C' and a lifting (the sequence splits in each dimension) of those in C'', then $t(C) = t(C') + t(C'')$.

PROOF

$$\begin{array}{ccccccccc} 0 & \to & C'_{r+1} & \to & C_{r+1} & \to & C''_{r+1} & \to & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \to & C'_r & \to & C_r & \to & C''_r & \to & 0 \end{array}$$

Let A be the matrix of α with respect to the given bases, and similarly A', A''. Then A must be of the form $\begin{pmatrix} A' & 0 \\ B & A'' \end{pmatrix}$ so that $t(A) = t(A') + t(A'')$.

that is by definition $t(C) = t(C') + t(C'')$.

ADDENDUM Since, performing Whitehead's folding simultaneously for C', C and C'', the 'exactness' property of the bases is preserved, the extension of 4.2 to arbitrary acyclic complexes of based free R-modules follows at once.

Let K be a complex rel N and the inclusion of N in K be a homotopy equivalence, then the torsion of K rel N, t(K,N) is defined to be the $[\pm \Pi]$-torsion of the acyclic based Z $[\Pi]$-complex $C_*(\tilde{K},\tilde{N};Z)$, where Π is the fundamental group of K and N, and the bases in C_* are those given by the cells of K-N. If t(K,N) = 0 then the inclusion of N in K is called a simple homotopy equivalence, and K and N are said to be of the same simple homotopy type. More generally if f : K $\longrightarrow$ L is a homotopy equivalence then its torsion t(f) is defined to be that of (C_f,K) where C_f is the mapping cylinder of f. ($C_f = L \cup K \times I / \{f(k) \sim (k,1)\}$ where $k \in K$ and K is embedded as $K \times [0]$). Here K and L must be cell complexes and the cells of C_f are those of K and L together with an (r+1)-cell for each r-cell of K. Again f is a simple homotopy equivalence when t(f) = 0. It then induces a simple equivalence of the corresponding based chain complexes. There are similar definitions when K and L are both complexes rel N and f a homotopy equivalence of (K,N) with (L,N).

Writing M = DP rel Q when M and P are complexes rel Q such that one is obtained from the other by a sequence of elementary expansions and contractions (see below and diagram 10) which do not affect Q, we note that $L = DC_f$ and that if i is the inclusion of a subcomplex N in the complex K then $C_i = D(K \times I) = D(K \times [0])$ rel $(N \times [0])$. From the latter and the next lemma it will follow that t(i) = t(K,N).

LEMMA 4.3 If K and M are complexes rel N such that M = DK rel N and t(K,N) is defined, then so is t(M,N) and is equal to t(K,N).

PROOF It is clear that the inclusion of N in M is a homotopy equivalence so that t(M,N) is defined. Proceeding by induction on the number of deformations in D it is sufficient to assume that $K = M - \mathring{e}^n - \mathring{e}^{n-1}$, where the e's are cells which do not meet N (we prove 4.3 for a cellular collapse, a simplicial collapse is a special case of this and the map which attaches e^n in M maps the 'southern hemisphere' into K and the 'northern hemisphere' homeomorphically onto e^{n-1}.

e^n meets K only along $e^n - \mathring{e}^{n-1}$

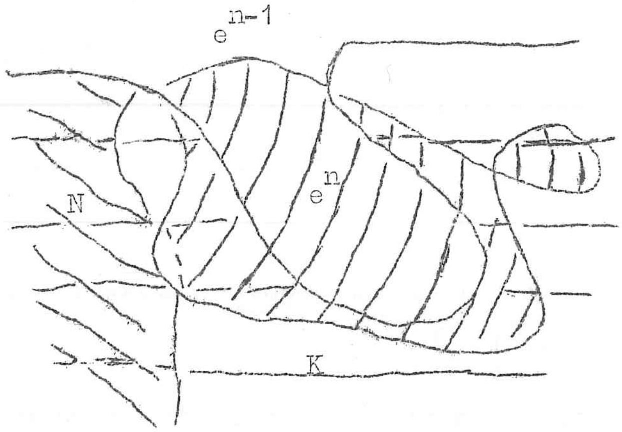


Diagram 10

$C_\pi(\tilde{K},\tilde{N})$ differs from $C_\pi(\tilde{M},\tilde{N})$ only in that the latter has generators in dimensions n, n-1 corresponding to $\tilde{e}^n$, $\tilde{e}^{n-1}$. It is clear that, with a suitable choice for the names of these cells we have

$$b(e^n,1) = (e^{n-1},1) + \Sigma = e'$$

where Σ is a linear combination with coefficients from $Z[\Pi]$ of the basis elements $(e_1,1)$, ... $(e_k,1)$ corresponding to the (n-1)-cells of K-N. Clearly the basis of $C_{n-1}(\tilde{M},\tilde{N})$ may be changed to e', $(e_1,1)$,... $(e_k,1)$ by moves which are $Z[\Pi]$-torsion-free (i.e. we are allowed to add $\pm\pi(e_s,1)$ to $(e^{n-1},1)$ for any π in Π and repeat this as often as necessary). Then $C_\pi(\tilde{M},\tilde{N})$ will differ from $C_\pi(\tilde{K},\tilde{N})$ by the appearance in the former of basis elements e', $(e^n,1)$ such that $b(e^n,1) = e'$, an addition which does not affect the torsion. Thus $t(M,N) = t(K,N)$.

COROLLARY 4.4 If K is the complex rel N associated with a handle decomposition of M on N then $t(K,N) = t(M,N)$ if either is defined.

REMARK In writing t(M,N) rather than t($\mathcal{D}$) we are assuming Whitehead's results that t is invariant under subdivisions ([13], see also [7]) and that a differential manifold has a unique combinatorial structure ([14]), so that t is a diffeomorphism invariant. Lemma 4.3 could also have been deduced from the work of Cockcroft on the relation between Reidemeister torsion and simple homotopy type torsion, for from theorem 13 of [13] there is a homotopy equivalence $\theta : M,N \longrightarrow K,N$ such that $t(\theta) = 0$ hence from [2] $t(K,N) = t(M,N)$.

5. CANCELLATION OF HANDLES

Throughout this paragraph M will denote a manifold with a handle decomposition on N with handles of types r and r+1 only, and (except when r=0) all the handles will be attached to the same component Q of bN. K will denote the associated complex rel N. It will be shown that the 'superfluous' handles of type r may be removed provided certain dimensional restrictions are satisfied and either $t(M,N) = 0$ or an equal number of trivial (r+2)-handles are added. The cases r=0,1 are special.

LEMMA 5.1 Let r=0 and N meet p connected components of M which has q components, then all except q - p 0-handles may be cancelled.

PROOF. Whenever one end of a 1-handle rests on a 0-handle they form a complementary pair and may be cancelled. Addition of an 0-handle increases the number of components by 1, and addition of a 1-handle either reduces the number by 1 or does not affect it. The result follows by counting components.

COROLLARY 5.2 If M is connected then all the 0-handles may be removed when N is non-empty, and all except one when N is empty.

For the case r=1 we use the Tietze transformations of the presentation of a group. (See for example Kurosh [5]). Let the group G be given by a finite set of generators a and a finite set of relations $\beta(a)=1$ in terms of these generators. It is clear that the following transformations, and their inverses, will give different presentations of the same group G.

(A) Introduce a new generator c and a relation of the form $c^{-1}\gamma(a) = 1$.

(B) Remove a relation which is a consequence of, that is a product of conjugates of, the others.

LEMMA 5.3 (Tietze, Kurosh) If $G = [a,b \,/\, \delta(a) = 1 = \lambda(a,b)]$ and $G = [a,c \,/\, \delta(a) = 1 = \mu(a,c)]$ are two presentations of the group G each containing a presentation of the subgroup $F = [a \,/\, \delta(a) = 1]$, then each may be obtained from the other by Tietze transformations, (A), (B) or their inverses (A'), (B'), which do not involve the presentation of F, that is which do not remove a generator from a nor a relation from $\delta(a)$.

PROOF. For each generator c there must be a relation $c^{-1}\gamma(a,b) = 1$, and for each b a relation $b^{-1}\beta(a,c) = 1$.

Thus $G = [a,b \,/\, \delta(a) = 1 = \lambda(a,b)]$

$= [a,b,c \,/\, \delta(a) = \lambda(a,b) = 1 = c^{-1}\gamma(a,b)]$

$= [a,b,c \,/\, \delta(a) = \lambda(a,b) = c^{-1}\gamma(a,b) = 1 = \mu(a,c) = b^{-1}\beta(a,c)]$ where, in the last presentation, each relation of the last two sets must be a consequence of those in the first three sets. Thus the presentations differ by transformations which clearly do not affect a or $\delta(a)$, and similar transformations will lead to $G = [a,c \,/\, \delta(a) = 1 = \lambda(a,b)]$.

To relate this to the geometry assume that the handles are added to the same component Q of bN so the 1-handles are trivial, and that n, the dimension of M and N, is at least 5 so the handles are attached to manifolds in which, since they have dimension at least 4, homotopically embedded circles are isotopic (Haefliger [4]). Assume further that the inclusion of Q in N induces an isomorphism of the fundamental groups. The handles express the group $G = \pi_1(M)$ as follows. Starting from any presentation $F = [a \,/\, \delta(a) = 1]$ of $F = \pi_1(N) = \pi_1(Q)$ the 1-handles, being trivial, introduce new generators b. If the a-sphere, a circle, of a 2-handle carries in N+(1-handles) the element $\lambda(a,b)$ then this 2-handle introduces the relation $\lambda(a,b) = 1$, giving the presentation $G = \left[a,b \,/\, \delta(a) = 1 = \lambda(a,b) \right]$.

The transformation (A) is realised by a complementary pair of 1- and 2-handles; since $\pi_1(Q) = \pi_1(N)$ the generators a of F are carried by circles in Q and as the generators introduced by 1-handles may clearly be carried in the boundary (c.f. 1.5), the element $\gamma(a,b)$ may be carried by a circle C_1 in b(N+(1-handles)). Add a trivial 1-handle and let a circle C_2 in the boundary carry the new generator c. Then the embedded circle $C_1 \# -C_2$ may be used to attach a 2-handle provided it has trivial normal bundle, that is provided the 1-handle is chosen so that C_2 has the same orientation as C_1. This handle is complementary to the 1-handle and introduces the relation $c^{-1}\gamma(a,b) = 1$. The reverse transformation is also possible provided we know that the generator corresponds to a 1-handle and not to a generator of the set a. For then the relation must correspond to a 2-handle whose a-sphere goes just once round the 1-handle, so they form a complementary pair and may be removed.

If β_1, β_2 are the elements of $\pi_1(Q) = F$ carried by the a-spheres of two 2-handles, then by lemma 1.4 these may be replaced by handles whose a-spheres carry $\beta_1\beta_2^{\pm x}$ and β_2 for any x in F. To see this give the a-spheres S_1, S_2 (circles) local base points P_i^*, as described in 2.1 for the general case, and pull P_2^* around a path representing x in Q and back into the base disc D^* before adding S_2 to S_1 as in 1.4, performing this addition (i.e. the path α of 1.4) within D^* between the base points P_i^*. Thus if the relation β, carried by a handle, is a consequence of the others, including perhaps some from $\delta(a)$, then repetition of the above procedure and moving the a-sphere across an immersed disc which spans the relation δ (a) allows the handle carrying β to be replaced by a trivial 2-handle. This may be regarded as carrying the empty relation 1 = 1, and so we realise (B) in the sense of replacing a relation, which is carried by a handle and is the consequence of the other relations, by the empty relation. Similarly we may realise (B') by introducing a trivial 2-handle and moving its a-sphere until it carries the required relation. To realise (B') with an equivalent decomposition we must also introduce a complementary 3-handle.

From lemma 5.3 and the above discussion follows

LEMMA 5.4 If M^n has a handle decomposition on N^n, with only 1- and 2-handles which are attached to the component Q of bN, if $\pi_1(Q) \cong \pi_1(N) = F$ by inclusion, if $n > 4$ and if $\pi_1(M) = G = [a,c \,/\, \delta(a) = \mu(a,c) = 1]$ is a presentation of G containing one of $iF = [a \,/\, \delta(a) = 1]$, where $i : F \longrightarrow G$ is the homomorphism induced by inclusion then the given decomposition is equivalent to one in which the 1-handles carry the generators c, 2-handles are trivial except for one to carry each relation μ and one to carry each of a set of generators of the kernel of i, and the remaining handles are of type 3.

PROOF. Let $F = [a \,/\, \delta'(a) = 1]$ and let the elements $\delta''(a)$ generate the kernel of i. Then the $\delta''(a)$ must be a consequence of the relations introduced by the 2-handles and so, introducing complementary pairs of 2- and 3-handles into the decomposition to realise (B'), each $\delta''(a)$ may be carried by a 2-handle attached to Q. Now if $\delta = (\delta', \delta'')$ we have a presentation $iF = [a/ \delta(a) = 1]$, which without loss of generality we may take for that in the statement of the lemma. Let the 1-handles of this decomposition, equivalent to the first, carry generators b, and the remaining 2-handles carry relations $\lambda(a,b) = 1$, then we have presentation $G = \left[a,b \,/\, \delta(a) = 1 = \lambda(a,b) \right]$ and may apply 5.3 to this and the given one. As the transformations used in 5.3 which remove generators or relations only remove some of those which correspond to handles, all these transformations may be realised by handle moves, which will lead to the required decomposition equivalent to the first.

COROLLARY 5.5 If M and N are as in 5.4 with $\pi_1(M) \cong \pi_1(N)$ by inclusion then the given decomposition is equivalent to one with no 1-handles, only trivial 2-handles and the same number of 3-handles as there were 1-handles.

PROOF. Use the presentation $G = \left[a \,/\, \delta(a) = 1 \right]$ in 5.4.

COROLLARY 5.6 If M_n has dimension at least 5 and is connected and $G = \left[a \,/\, \delta(a) = 1 \right]$ is a presentation of its fundamental group then M has a handle decomposition with one 0-handle, a 1-handle to carry each a, a 2-handle to carry each $\delta(a)$ and trivial 2-handles and the remaining handles of type at least 3.

PROOF. By Smale [10] M has a handle decomposition with the handles in order of type. Let M' be that part consisting of 0-, 1- and 2-handles. By 5.2 all the 0-handles except one, D^n, may be removed. Now apply 5.4 with D^n for N and F having the empty presentation to obtain an equivalent decomposition of M' with the 0-, 1- and 2-handles as described plus some 3-handles. Add on the original handles of types 3,...,n to reform M.

We now consider decompositions with handles of other indices.

LEMMA 5.7 If M^n has a decomposition on N with r- and (r+1)-handles added to a component Q of bN such that inclusion in N induces an isomorphism

$\pi_1(Q) \cong \pi_1(N)$, if $1 < r < n-3$, and

if the inclusion of N in M is a homotopy equivalence and $t(M,N) = 0$ then M is diffeomorphic to N.

PROOF. Let C_s denote $C_s(\tilde{M},\tilde{N})$ with base determined by the s-handles and $b : C_{r+1} \longrightarrow C_r$ be the boundary homomorphism. Then $t(b) = 0$, that is for some based free $Z[\Pi]$-module E, the homomorphism $(b, id) : C_{r+1} + E \longrightarrow C_r+E$ has matrix B, with respect to the given bases, which is a product of $[\pm\Pi]$--elementary matrices of the same order. Now the insertion into the decomposition of complementary pairs of r- and (r+1)-handles, in number equal to the rank of E, will change C_s to $C_s + E$ for $s = r, r+1$ extending b by the identity on E. Then by 3.5 further handle moves may be performed to obtain a new decomposition in which the boundary has matrix $B^{-1}B = I$. Thus if h_i^{r+1} and h_i^r, $i = 1,\dots,k$ are the new handles with corresponding base elements e_i^{r+1}, e_i^r then $be_i^{r+1} = e_i^r$ for each i when the handles are suitably ordered. By 3.1 this means that if S_a is the a-sphere of h_i^{r+1} and S_b the b-sphere of h_i^r then S_a meets S_b in a point P and pairs of points P_j, P_j' having opposite signs $\sigma_j = -\sigma_j'$ and equal characteristics $\pi_j = \pi_j'$. All the pairs of intersections P_j, P_j' may be removed by 2.2 since we are assuming $1 < r < n-3$ and then h_i^{r+1} and h_i^r form a complementary pair of handles and may be removed by 1.3. Repeating this for all i we obtain a decomposition of M on N with no handles, from which it is clear that $M \cong N + Q \times I \cong N$.

More generally we have

LEMMA 5.8 Let M^n have a decomposition $\mathcal{D}$ on $Q \times I$ with r- and (r+1)-handles, the former attached trivially when r=2, where $1 < r < n-2$ and let $C = C_*(\tilde{M},\tilde{Q})$ be the associated based $Z[\Pi]$-complex where $\Pi = \pi_1(M)$. Then, if

$f : C \dashrightarrow D$ is a simple isomorphism[‡] with a $Z[\Pi]$-complex D, there is an equivalent decomposition $\mathcal{E}$ with associated diffeomorphism $d : \mathcal{D} \dashrightarrow \mathcal{E}$, such that $\mathcal{E}$ has associated complex D and $d_* = f : C \dashrightarrow D$.

PROOF. Note that by lemma 1.5 $\pi_1(Q) \cong \pi_1(Q + \text{handles})$ for the handles concerned so that C is indeed a $Z[\Pi]$-complex. f being a simple isomorphism is in particular an isomorphism in each dimension so that D is also zero outside dimensions r, r+1. The lemma is an enlargement of the addendum to 3.4. Note that in that addendum the presence of r-handles was irrelevant; though they did serve, after the calculation in the proof of 3.4, to identify the matrices of the simple isomorphism of the associated chain complexes, it was already clear, from the definitions of the associated chain complexes and the diffeomorphism associated with a handle move, that these are the identity outside the dimension of the handle that is moved, that in this dimension it is $[\pm\Pi]$-elementary and that in this way all $[\pm\Pi]$- elementary matrices could be realised. It is also possible to realise such matrices in dimension two when the handles are attached trivially. (When they are not so attached the moves are still possible but the handles disturb the fundamental group). Thus we may in all cases find a sequence of moves, first of r-handles and then of (r+1)-handles such that if $d: \mathcal{D} \dashrightarrow \mathcal{E}$ is the composition of the associated diffeomorphisms then $\mathcal{E}$ has associated complex D and d induces $f : C \dashrightarrow D$.

COROLLARY 5.9 Let M, Q, $\mathcal{D}$, C be as in the lemma and let D be a $Z[\Pi]$-complex, zero outside dimensions r, r+1, and $f : C \dashrightarrow D$ a simple equivalence ($t(f) = 0$), then provided $r < n-3$ there is an equivalence $d : \mathcal{D} \dashrightarrow \mathcal{E}$ where $\mathcal{E}$ has associated complex D and d induces an equivalence chain homotopic to f.

‡ See footnote on p.30.

PROOF. By the result of Whitehead, mentioned in paragraph 4, f is chain homotopic to a composition $k \circ g \circ i$, where for collapsible B, B' i is the inclusion $C \longrightarrow C + B$, g a simple isomorphism $C + B \longrightarrow C' + B'$ and k the projection $C' + B' \longrightarrow C'$. B, B' may be assumed zero outside dimensions r, r+1 and may also be taken to be trivial elementary complexes of the form $0 \longrightarrow E \xrightarrow{id} E \longrightarrow 0$. For such a choice we shall find $d : \mathcal{D} \longrightarrow \mathcal{E}$ such that $d_* = kgi$. i is realised by the addition of the required number of complementary pairs of r- and (r+1)-handles and g is realised by 5.8. The realisation of k involves the cancellation of pairs of handles h_i^{r+1}, h_i^r for which the corresponding base elements satisfy $be_i^{r+1} = e_i^r$. This may be achieved as in the proof of 5.7.

ADDENDUM. The case $r = n-3$ may be included if we know that the duals (2-handles) of any (n-2)-handles are attached trivially to the 'top' boundary component of M, and $n > 5$.

PROOF It remains to cancel certain pairs of (n-3)- and (n-2)-handles. Add the other (n-3)-handles first and let the top boundary of the result be Q'. Then add the remaining (n-3)-handles and their paired (n-2)-handles to $Q' \times I$. By the choice of handles the boundary in this subdecomposition has the identity matrix and hence, by 3.6, so has the boundary in the dual decomposition of 2- and 3-handles. Now since $2 < n-3$ lemma 2.2 is available for this decomposition and we cancel the handles as before.

6. THEOREMS

THEOREM 6.1 Let Q, Q' be the two connected components of bM^n where $n > 4$, M is connected compact and smooth and the inclusions of Q, Q' in M induce isomorphisms of fundamental groups. Then M has a decomposition $\mathcal{D}$ on Q without 0-, 1-, (n-1)- or n-handles. If $C = C_*(\tilde{M}, \tilde{Q})$ is the corresponding based $Z[\Pi]$-complex and $f : C \dashrightarrow D$ a simple equivalence where D is zero in dimensions < 2 and $> n-2$, then if $n \geq 5$ there is an equivalence $d : \mathcal{D} \dashrightarrow \mathcal{E}$ where $\mathcal{E}$ has associated complex D and d_* is chain homotopic to f.

PROOF First M, being smooth and compact has a normalised decomposition on Q ([9], [10]). Then by 5.2 and 5.5 we can find an equivalent decomposition without 0- or 1-handles and, applying the same results to the dual decomposition, without (n-1)- or n-handles. This is $\mathcal{D}$. The rest of the proof is by repeated applications of 5.8 and 5.9. Note that since $\pi_1(Q) \cong \pi_1(M)$ by inclusion and $\mathcal{D}$ has no 1-handles all 2-handles must be trivial, and similarly the same is true in the dual $\mathcal{D}^*$, so that the lemmas are applicable. Choose collapsible B, B' and a simple isomorphism $g : C + B \dashrightarrow C' + B'$ such that f is homotopic to kgi in the usual notation and where as usual B, B' are direct sums of trivial elementary complexes zero outside dimensions 2 through n-2. $i : C \dashrightarrow C + B$ is realised by inserting pairs of complementary handles. Recall that, as remarked during the proof of 5.8, if $d : \mathcal{D}_1 \dashrightarrow \mathcal{D}_2$ is the diffeomorphism associated with a move of an r-handle, then the induced equivalence of associated $Z[\Pi]$ complexes is the identity in dimensions other than r. Similarly when a pair of handles is cancelled we may assume the induced equivalence is the 'identity' on all cells other than those cancelled.

Thus g may be realised by applying 5.8 to each of the subdecompositions consisting only of the r-handles. k is realised similarly by applying 5.9 to each of the subdecompositions consisting of the r- and (r+1)-handles.

REMARK 1 This includes Smale's more specific result for the case when Q, Q' and M are simply-connected. For then $\tilde{M} = M$, $\tilde{Q} = Q$ and so $C = C_{\pi}(M,Q)$. C is now a complex of Z-modules and by Z- ($= Z[1]$-) elementary moves may be changed into any other with the same homology, and in particular into the one with the minimum number of generators consistent with its homology. By the theorem this simple equivalence may be realised by handle moves giving a decomposition with the minimum possible number of handles.

REMARK 2 When the fundamental groups of the boundary components are not isomorphic by inclusion with that of M then a limited result may be obtained combining 6.1 with 5.4; after removing from each end subdecompositions, provided by 5.4, which 'express' the respective inclusion homomorphisms we are left with a decomposition to which 6.1 applies.

COROLLARY 6.2 If bM^n, $n > 4$, has connected components Q, Q' such that the inclusion of Q in M is a homotopy equivalence and that of Q' induces an isomorphism of fundamental groups, then any decomposition of M on Q is equivalent to one with only r- and (r+1)-handles for any chosen r such that $1 < r < n-2$.

PROOF Note that the decomposition $\mathcal{D}$ obtained in the theorem could be chosen equivalent to any given decomposition. When n = 5 this is sufficient for the corollary. For the general case note that C, the complex associated with $\mathcal{D}$, is acyclic. The result then follows from the observation that, while defining the torsion of such a complex in paragraph 4, we in effect defined a simple equivalence with a complex which is non-zero in only two dimensions and by 6.1 this equivalence is realisable by handle moves.

ALITER For convenience and to clarify the ideas we give the details of the above proof, 'realising' Whitehead's folding procedure, using only the results of the first three paragraphs and the existence of the decomposition $\mathcal{D}$ without 0-, 1-, (n-1)- or n-handles. Let then e^r be a basis element, of the associated complex C, corresponding to a handle of lowest index. ($r < n-3$). Insert a pair of complementary (r+1)- and (r+2)-handles introducing new base elements f^{r+2}, f^{r+1}. Then $bf^{r+1} = 0$ and, since C is acyclic and $be^r = 0$, $e^r = b\sum_i \lambda_i e_i^{r+1}$ for some λ_i in $Z[\Pi]$ where e_i^{r+1} are the other base elements of C_{r+1}. Thus by a sequence of moves of the type described in 3.4(ii) we may move the handle corresponding to f^{r+1} to a position where the new element f_1 corresponding to it satisfies $bf_1 = e^r$. Now by 3.1 all the intersections except one of the corresponding a-sphere and b-sphere fall into pairs, the members of each pair having homotopic characteristic paths and opposite sign. By 2.2 the a-sphere may be moved until it has only one intersection with the b-sphere and then both handles may be removed by 1.3. There is now one fewer r-handle and clearly we may continue until there are only (n-3)- and (n-2)-handles. The boundary homomorphism b must now be an isomorphism and hence by 3.6 so is the boundary b^* in the dual decomposition on Q'. Thus we may repeat the argument in $\mathcal{D}^*$ and so replace the handles by r- and (r+1)-handles for any r we choose such that $1 < r < n-2$.

COROLLARY 6.3 (S-cobordism theorem - Mazur [6]). If $n > 5$ and bM^n has connected components Q, Q' such that the inclusion of Q in M is a simple homotopy equivalence (t(M,Q) is defined and is zero), and the inclusion of

Q' in M induces an isomorphism of fundamental groups, then M is diffeomorphic to $Q \times I$ and so Q and Q' are diffeomorphic.

PROOF Take the decomposition of 6.2 with say 2- and 3-handles and then apply 5.7 with $N = Q \times I$.

For the case when M has a decomposition on a submanifold other than a boundary component these results become

COROLLARY 6.4 If $n > 4$, the inclusion of the submanifold N^n in M^n is a homotopy equivalence, $\mathcal{D}$ is a decomposition of M on N with handles attached to a component Q of bN such that $\pi_1(Q) \cong \pi_1(N)$ by inclusion and if the 'top' component Q' of bM satisfies $\pi_1(Q') \cong \pi_1(\overline{M-N})$ by inclusion then

(i) There is an equivalent decomposition with handles only of types r and $(r+1)$ for any r such that $1 < r < n-2$.

(ii) If $n > 5$ and the inclusion of N in M is a simple homotopy equivalence then M is diffeomorphic to N.

PROOF As above, except that the 'dual' $\mathcal{D}^*$ is now a decomposition of $\overline{M-N}$ on Q' so that to remove $(n-1)$- and n-handles we need to know that $\pi_1(Q') \cong \pi_1(\overline{M-N})$ by inclusion. When $n > 5$ proceed as above until there are only $(n-3)$- and $(n-2)$-handles. Then by 1.5 $\pi_1(Q) \cong \pi_1(\overline{M-N})$ by inclusion and, since $\pi_1(Q) \cong \pi_1(N)$ by the inclusion, b which is an isomorphism in $C_*(\widetilde{M},\widetilde{N})$ is also an isomorphism in $C_*(\widetilde{\overline{M-N}},\widetilde{Q})$ and we may continue as before in $\mathcal{D}^*$. When there are only say 2- and 3-handles 5.7 applies in $\mathcal{D}$.

COROLLARY 6.5 If W^n is a compact connected manifold and M^k a closed submanifold whose inclusion is a simple homotopy equivalence, if M is disjoint from bW, $\pi_1(bW) \cong \pi_1(W-M)$ by inclusion $k < n-2$ and $n > 5$ then W is diffeomorphic to any tubular neighborhood T of M in its interior.

PROOF T clearly collapses onto M and so by 4.3 the inclusion of M in T is a simple homotopy equivalence. $\pi_1(T) \cong \pi_1(M) \cong \pi_1(W)$ and there is an exact sequence

$$0 \longrightarrow C_*(\tilde{T},\tilde{M}) \longrightarrow C_*(\tilde{W},\tilde{M}) \longrightarrow C_*(\tilde{W},\tilde{T}) \longrightarrow 0.$$

Thus, as $\pi_*(W,T) \cong \pi_*(\tilde{W},\tilde{T}) \cong H_*(\tilde{W},\tilde{T}) = 0$, the inclusion of T in W is a homotopy equivalence. Since the cells of W-M are those of W-T together with those of T-M, the sequence preserves bases as required in 4.2 and so by its addendum $t(W,T) = 0$. Now $\pi_1(bW) \cong \pi_1(W-M) \cong \pi_1(W-T)$, and also since bT is an (n-k-1)-sphere bundle over M $\pi_1(bT) \cong \pi_1(M) \cong \pi_1(T)$. Then 6.4(ii) gives the required diffeomorphism between W and T.

ADDENDUM The corollary is valid for $k = n-2$ if it is known that the projection induces an isomorphism $\pi_1(bT) \cong \pi_1(M)$.

COROLLARY 6.6 If $f : M_1^n \dashrightarrow M_2^n$ is a simple tangential homotopy equivalence, that is a simple homotopy equivalence such that $\tau M_1 = f^*(\tau M_2)$ where τ denotes the tangent bundle, then $M_1 \times D^k \cong M_2 \times D^k$ for $k > n > 2$.

PROOF Embed M_1 in $M_2 \times D^k$ approximating $f(M_1)$. Then M_1 has trivial normal bundle $M_1 \times D^k$ and its inclusion in $M_2 \times D^k$ is a simple homotopy equivalence. The result follows from 6.5 since $\pi_1(M_2 \times S^{k-1}) \cong \pi_1(M_2) \cong \pi_1(M_2 \times D^k) \cong \pi_1(M_2 \times D^k - M_1)$, the last isomorphism being because M_1 has codimension $k > 2$.

ADDENDUM If $2k > n+2$ then any k-cell bundle over M_2^n is diffeomorphic to one over M_2^n, **provided that** $k > 2$.

PROOF Since $2(n+k) > 3n+2$, M_1 may be embedded, approximating $f(M_1)$ in the given bundle over M_2 using Haefliger's theorem ([4]) and the proof proceeds as before except that we do not now identify the normal bundle of M_1.

COROLLARY 6.7 If $f : W_1^n \dashrightarrow W_2^n$ is a simple tangential homotopy equivalence, if $\pi_k(W_1, bW_1) = 0$ for all $k < n-m$ where $n > 5$ and $n > 2m$ and if $\pi_1(bW_2) \cong \pi_1(W_2)$ by inclusion then there is a diffeomorphism $g : W_1 \dashrightarrow W_2$ homotopic to f.

PROOF By the hypotheses $\pi_1(bW_1) \cong \pi_1(W_1)$ and by the usual isomorphisms $H_k(\tilde{W}_1, b\tilde{W}_1) \cong \pi_k(W_1, bW_1)$ when zero, i.e. for $k < n-m$. Thus, taking a decomposition of W_1 on bW_1 without 0- or 1-handles, if the first handles to appear are of type k for $k < n-m$ then in the associated complex $b : C_{k+1} \dashrightarrow C_k$ is epimorphic and so the k-handles may be removed in the usual manner. (c.f. 6.2). After removing all such handles the dual is a decomposition of W_1 without handles of index greater than m. We may now proceed as did Smale in [11] (Theorem 7.1). Define an embedding $f' : W_1 \dashrightarrow \overset{\circ}{W}_2$, homotopic to f, handle by handle. Assuming all handles of type $< k$ already embedded, to extend the embedding over $h^k = D^k \times D^{n-k}$ $(k \leqslant m)$ first approximate $f(D^k \times 0)$ by an embedding which extends $f'(S^{k-1} \times 0)$ and avoids the handles already embedded. This is possible since these handles have essential dimension $\leqslant k$ and $n > 2m \geqslant 2k$. Now f, and so also f' which is homotopic to it, is a tangential equivalence and this ensures that f' may be extended over the handle. The completed embedding $f'(W_1)$ is included by a simple homotopy equivalence in $\overset{\circ}{W}_2$ and also $\pi_1(bW_2) \cong \pi_1(W_2) \cong \pi_1(W_2 - f'W_1)$ since $f'W_1$ has essential codimension $n-m > 2$. Thus by 6.4(ii) $W_2 - f'\overset{\circ}{W}_1 \cong bW_1 \times I$ and there is a diffeomorphism $g : W_1 \cong W_2$ homotopic to f' and so to f.

COROLLARY 6.8 Let M^{2k+1} be a closed manifold with $k > 2$, then there is a compact manifold W^{2k+1} and a diffeomorphism $h : bW \dashrightarrow bW$ such that $M = W + h(W)$.

PROOF Let P_1 be the union of all handles up to and including those of index k in a decomposition of M. P_1 has essential dimension k and so may be moved slightly to a homotopic position P_2 disjoint from it. By Haefliger's theorems ([4]) P_2 may be embedded and also, writing $S_i = bP_i$ and $R = \overline{M-P_1-P_2}$, there is an isotopy of M, taking P_1 to P_2, ~~which inter-changes them~~ (this uses the fact that $\pi_1(P_i) \dashrightarrow \pi_1(M)$ is an epimorphism) and so inducing a diffeomorphism $h : M, P_1 \dashrightarrow M, P_2$. Now if $Q_i = \overline{M-P_i}$ we have by construction $H_s(\tilde{M}, \tilde{P}_i) = 0 = H_s(\tilde{M}, \tilde{Q}_i)$ for $s \leqslant k$ and it follows from the homology sequence that $H_s(\tilde{Q}_2,\tilde{P}_1) = 0$ and by excision that $H_s(\tilde{R},\tilde{S}_1) = 0$ for $s < k$. Thus R has a decomposition $\mathcal{D}$ on S_1 without handles of index less than k, and by a similar argument in the dual without handles of index $>$ k+1. But by 1.5 $\pi_{k-1}(P_i) \cong \pi_{k-1}(S_i)$ by inclusion and the former is $\pi_{k-1}(M)$ so the k-handles must be attached trivially to S_1 (and similarly, in the dual, to S_2). Thus the k-handles of $\mathcal{D}$ in effect add a handlebody $H_1 \in \mathcal{H}(2k+1,r,k)$ to $S_1 \times I$ and similarly the k-handles of $\mathcal{D}^*$ add H_2 to $S_2 \times I$. It is clear that $H_1 \cong H_2$ so that the diffeomorphism of P_1 with P_2 extends to one of P_1+H_1 with P_2+H_2 and we may take these for our two copies of W.

The above results are generalisations of the theorems of Smale appearing in sections 4 through 8 of [11]. 6.5 is similar to Mazur's theorem on the uniqueness of 'simple neighborhoods', [6]; Mazur calls W a <u>simple neighborhood</u> of M when $\pi_1(bW) \cong \pi_1(W-M)$ and the inclusion of M in W is a simple homotopy equivalence. The following is related to the duality theorem of Milnor in [8].

THEOREM 6.9 If $n > 4$ and bM^n has connected components Q, Q' such that $t(M,Q)$ is defined and $\pi_1(Q') \cong \pi_1(M)$ by inclusion, then $t(M,Q')$ is defined and is equal to $(-1)^{n-1}\,\bar{t}'(M,Q)$.

PROOF $\bar{t}$ is t followed by the automorphism of $W_{[\pm\Gamma]}(Z[\Gamma])$ determined by mapping π to π^{-1} for each π in Γ. Since the inclusion of Q in M is a homotopy equivalence 6.2 provides a decomposition $\mathcal{D}$ of M on Q with only r- and (r+1)-handles. By 3.6 the boundary homomorphisms in the based chain complexes associated with $\mathcal{D}$ and $\mathcal{D}^*$ have matrices B and $\pm\bar{B}'$ respectively where ' denotes the transposed matrix. Now A is an invertible $Z[\Gamma]$-matrix if and only if $\bar{A}'$ is, and if and only if $\bar{A}$ is. So if $t(M,Q) = (-1)^r t(B)$ is defined so is $t(M,Q') = (-1)^{n-r-1} t(\bar{B}') = (-1)^{n-1}\bar{t}'(M,Q)$.

REFERENCES

[1] J. Cerf, 'Topologie de certains espaces de plongements', Bull.Soc.Math. France, 89 (1961) 227-380.

[2] W. H. Cockroft, 'Simple homotopy type torsion and the Reidemeister-Franz torsion', Toplogy 1 (1962) 143-150.

[3] A. Douady, Exposés 1-3, Séminaire Henri Cartan (Topologie différentielle) 1961/2.

[4] A. Haefliger, 'Plongements différentiables de variétés dans variétés', Comment.Math.Helv., 36 (1961) 575-590.

[5] A. G. Kurosh, Theory of groups, translated by K. A. Hirsch. Chelsea Publishing Company, 1956.

[6] B. Mazur, Differential topology from the point of view of simple homotopy theory, I.H.E.S., Publications Mathématiques No. 15.

[7] J. Milnor, 'Two complexes which are homeomorphic but combinatorially distinct', Ann. of Math., 74 (1961) 575-590.

[8] J. Milnor, 'A duality theorem for Reidemeister torsion', Ann. of Math., 74 (1961) 137-147.

[9] S. Smale, 'On gradient dynamical systems', Ann. of Math., 74(1961) 199-206.

[10] S. Smale, 'Generalised Poincaré's conjecture in dimensions greater than four', Ann. of Math., 74 (1961) 391-406.

[11] S. Smale, 'The structure of manifolds', Amer. J. Math., 84 (1962) 387-399.

[12] C. T. C. Wall, Differential topology notes, mimeographed, Cambridge, 1963-4.

[13] J. H. C. Whitehead, 'Simple homotopy types', Amer. J. Math., 72 (1950) 1-57.

[14] J. H. C. Whitehead, 'On C^1-complexes', Ann. of Math., 41 (1940) 809-832.

[15] H. Whitney, 'The self-intersections of a smooth n-manifold in 2n-space', Ann. of Math., 45 (1944) 220-246.